Parallel Construction of Independent Spanning Trees on Enhanced Hypercubes

Jinn-Shyong Yang, Jou-Ming Chang, Kung-Jui Pai, and Hung-Chang Chan

Appendix A
Proofs of Claims in Theorem 1.

In the following proofs, we assume that $H_n^1(x) = \{f_{s-1}, f_{s-2}, \ldots, f_0\}$ with $f_{s-1} > f_{s-2} > \cdots > f_0$ and $H_n^0(x) = \{g_{t-1}, g_{t-2}, \ldots, g_0\}$ with $g_{t-1} > g_{t-2} > \cdots > g_0$, where all index arithmetics of f_p are taken modulo s, and all index arithmetics of g_q are taken modulo t.

Claim 1.1. *If x matches one of the conditions in Eqs.* (1), (2), (3), (4), (5), (6) (*or* (6′)), *then there is a unique path that connects x and 0 in T_i.*

Proof. Let $j = \text{NEXT}_x(i)$ and suppose $x \neq 0$. There are the following scenarios:

Case 1: $i \in H_{n-k}^1(x)$, $|H_{n-k}^1(x)| \leqslant (n-k)/2 + \lambda$ and $H_i^1(x) = \emptyset$ (cf. Eq. (5)). In this case, $j = \max H_n^1(x) = f_{s-1}$. Since $x_j = 1$, x is adjacent to $x - 2^j$ in T_i. Let $y = x - 2^j$. Clearly, $H_n^1(y) = \{f_{s-2}, f_{s-3}, \ldots, f_0\}$. Note that $|H_{n-k}^1(y)| \leqslant |H_{n-k}^1(x)| \leqslant (n-k)/2 + \lambda$ and $H_i^1(y) = H_i^1(x) = \emptyset$. Thus, if $H_n^1(y) = \emptyset$, then $y = 0$. In this case, $x \xrightarrow{-2^j} 0$ is the desired path connecting x and 0 in T_i. Otherwise, y is still in the situation of Case 1. By the same argument we know that y is adjacent to the node $y - 2^{j'}$ in T_i, where $j' = \text{NEXT}_y(i) = f_{s-2}$. Repeat this process until the path passes through the node 2^{f_0} in T_i. Therefore, we can find the following unique path that connects x and 0 in T_i: $P : x \xrightarrow{-2^{f_{s-1}}} (x - 2^{f_{s-1}}) \xrightarrow{-2^{f_{s-2}}} (x - 2^{f_{s-1}} - 2^{f_{s-2}}) \xrightarrow{-2^{f_{s-3}}} \cdots \xrightarrow{-2^{f_1}} (2^{f_0}) \xrightarrow{-2^{f_0}} 0$.

Case 2: $i \in H_{n-k}^1(x)$, $|H_{n-k}^1(x)| \leqslant (n-k)/2 + \lambda$ and $H_i^1(x) \neq \emptyset$ (cf. Eq. (4)). In this case, since $x_i = 1$ and $H_i^1(x) \neq \emptyset$, we suppose $i = f_p$ for some $0 < p \leqslant s-1$. By Eq. (4), $j = \max H_i^1(x) = f_{p-1}$. Since $x_j = 1$, x is adjacent to $x - 2^j$ in T_i. Let $y = x - 2^j$. Clearly, $H_i^1(y) = \{f_{p-2}, f_{p-3}, \ldots, f_0\}$ and $|H_{n-k}^1(y)| < |H_{n-k}^1(x)| \leqslant (n-k)/2 + \lambda$. Thus, if $H_i^1(y) \neq \emptyset$, we can repeat this process until the path passes through the node $w = x - \sum_{h=0}^{p-1} 2^{f_h}$. Let Q be the path described as follows: $Q : x \xrightarrow{-2^{f_{p-1}}} (x - 2^{f_{p-1}}) \xrightarrow{-2^{f_{p-2}}} (x - 2^{f_{p-1}} - 2^{f_{p-2}}) \xrightarrow{-2^{f_{p-3}}} \cdots \xrightarrow{-2^{f_0}} w$. Now, it is easy to check that $|H_{n-k}^1(w)| = |H_{n-k}^1(x)| - p \leqslant (n-k)/2 + \lambda$ and $H_i^1(w) = \emptyset$. Thus, w is in the situation of Case 1. Let P be the path connecting w and 0. Therefore, we can find the unique path $T_i[x, 0]$ by concatenating Q and P.

Case 3: $i \in H_{n-k}^0(x)$ and $|H_{n-k}^0(x)| > (n-k)/2$ (cf. Eq. (3)) In this case, we have $j = i$. Since $x_j = 0$, x is adjacent to $x + 2^j$ in T_i. Let $z = x + 2^j$. Clearly, $z_i = \bar{x}_i = 1$ and $i \in H_{n-k}^1(z)$. Also, $|H_{n-k}^0(x)| > (n-k)/2$ implies $|H_{n-k}^1(x)| < (n-k)/2$. Thus, we have $|H_{n-k}^1(z)| = |H_{n-k}^1(x)| + 1 \leqslant ((n-k)/2 - 0.5) + 1 \leqslant (n-k)/2 + \lambda$, where $\lambda = 0.5$ or $\lambda = 1$. This shows that z is in the situation of Case 1 for $H_i^0(z) \neq \emptyset$ or Case 2 for $H_i^0(z) = \emptyset$. No matter what the situation does z have, let P be the path connecting z and 0. Therefore, we can find the unique path $T_i[x, 0]$ by concatenating $x \xrightarrow{+2^j} z$ and P.

Case 4: $i \in H_{n-k}^0(x)$, $|H_{n-k}^0(x)| \leqslant (n-k)/2$, and $H_i^0(x) = \emptyset$ (cf. Eq. (2)). In this case, we have $j = *$. Let $z = x \oplus (2^{n-k} - 1)$ be the node adjacent to x in T_i. Since $i \in H_{n-k}^0(x)$, it implies $z_i = \bar{x}_i = 1$, and thus $i \in H_{n-k}^1(z)$. Clearly, $H_i^1(z) = H_i^0(x) = \emptyset$. Moreover, $|H_{n-k}^0(x)| \leqslant (n-k)/2$ implies $|H_{n-k}^1(z)| = |H_{n-k}^0(x)| < (n-k)/2 + \lambda$. This shows that z is in the situation of Case 1. Let P be the path connecting z and 0. Therefore, we can find the unique path $T_i[x, 0]$ by concatenating $x \xrightarrow{*} z$ and P.

Case 5: $i \in H_{n-k}^0(x)$, $|H_{n-k}^0(x)| \leqslant (n-k)/2$, and $H_i^0(x) \neq \emptyset$ (cf. Eq. (1)). In this case, since $x_i = 0$, we suppose $i = g_q$ for some $0 < q \leqslant t-1$. Since $H_i^0(x) \neq \emptyset$, by Eq. (1) we have $j = \max H_i^0(x) = g_{q-1}$. Since $x_j = 0$, x is adjacent to $x + 2^j$ in T_i. Let $y = x + 2^j$. Clearly, $H_i^0(y) = \{g_{q-2}, g_{q-3}, \ldots, g_0\}$ and $|H_{n-k}^0(y)| = |H_{n-k}^0(x)| - 1 \leqslant (n-k)/2$. Thus, if $H_i^0(y) \neq \emptyset$, we can repeat this process until the path passes through the node $w = x + \sum_{h=0}^{q-1} 2^{g_h}$. Let Q be the path described as follows: $Q : x \xrightarrow{+2^{g_{q-1}}} (x + 2^{g_{q-1}}) \xrightarrow{+2^{g_{q-2}}} (x + 2^{g_{q-1}} + 2^{g_{q-2}}) \xrightarrow{+2^{g_{q-3}}} \cdots \xrightarrow{+2^{g_0}} w$. Now, it is easy to check that $H_n^0(w) = \{g_{t-1}, g_{t-2}, \ldots, g_q\}$ and $H_i^0(w) = \emptyset$. Thus, $w_i = x_i = 0$ and $i \in H_{n-k}^0(w)$. Since $|H_{n-k}^0(w)| = |H_{n-k}^0(x)| - q \leqslant (n-k)/2$ and $H_i^0(w) = \emptyset$, w is in the situation of Case 4. Let P be the path connecting w and 0. Therefore, we can find the unique path $T_i[x, 0]$ by concatenating Q and P.

Case 6: $i \in H_{n-k}^1(x)$ and $|H_{n-k}^1(x)| > (n-k)/2 + \lambda$ (cf. Eq. (6)). In this case, we have $n - k > 2$ and $j = i$.

Since $x_j = 1$, x is adjacent to $x - 2^j$ in T_i. Let $z = x-2^j$. Clearly, $z_i = \bar{x}_i = 0$ and $i \in H^0_{n-k}(z)$. Moreover, $|H^1_{n-k}(x)| > (n-k)/2 + \lambda$ implies $|H^0_{n-k}(x)| < (n-k)/2 - \lambda$. Thus, we have $|H^0_{n-k}(z)| = |H^0_{n-k}(x)| + 1 \leqslant ((n-k)/2-\lambda-0.5)+1 \leqslant (n-k)/2$. This shows that z is in the situation of Case 4 for $H^0_i(z) = \emptyset$ or Case 5 for $H^0_i(z) \neq \emptyset$. No matter what the situation does z have, let P be the path that connects z and 0 in T_i. Thus, we obtain the unique path $T_i[x, 0]$ by concatenating $x \xrightarrow{-2^j} z$ and P.

Case 7: $i \in H^1_2(x)$ and $|H^1_2(x)| = 2$ (cf. Eq. (6′)). In this case, we have $n - k = 2$ and $\lambda = 0.5$. Since $x_0 = x_1 = 1$ and either $i = 1$ or $i = 0$, by Eq. (6′) we have $j = \max H^1_{2-i} = i \oplus 1$. Since $x_j = 1$, x is adjacent to $x-2^j$ in T_i. Let $z = x-2^j$. Clearly, $z_i = x_i = 1$ and $i \in H^1_{n-k}(z)$. Moreover, $|H^1_{n-k}(z)| = |H^1_{n-k}(x)| - 1 = 1 < (n-k)/2 + \lambda$ and $H^1_i(z) = \emptyset$. This shows that z is in the situation of Case 1. Let P be the path that connects z and 0 in T_i. Thus, we obtain the unique path $T_i[x, 0]$ by concatenating $x \xrightarrow{-2^j} z$ and P.

As a result, this completes the proof. □

Claim 1.2. *If x matches one of the conditions in Eqs.* (7), (8), (9), (10), (11), (12), *then there is a unique path that connects x and 0 in T_i.*

Proof. Let $j = \text{NEXT}_x(i)$ and suppose $x \neq 0$. There are the following scenarios:

Case 1: $i \in H^1_{n,n-k}(x)$ and $H^1_i(x) = \emptyset$ (cf. Eq. (8)). In this case, we have $j = \max H^1_n(x) = f_{s-1}$. Since $H^1_i(x) = \emptyset$, a proof similar to that of Case 1 in Claim 1.1 shows that there is a unique path connecting x and 0 in T_i.

Case 2: $i \in H^1_{n,n-k}(x)$, $H^1_{i,n-k}(x) = \emptyset$ and $0 < |H^1_{n-k}(x)| \leqslant (n-k)/2 + \lambda$ (cf. Eq. (10)). In this case, since $i \in H^1_{n,n-k}(x)$ and $H^1_{n-k}(x) \neq \emptyset$, by Eq. (10) we suppose $j = \max H^1_{n-k}(x) = f_p$ for some $0 \leqslant p < s-1$. A proof similar to that of Case 2 in Claim 1.1 shows that there is a unique path connecting x and 0 in T_i.

Case 3: $i \in H^1_{n,n-k}(x)$, $H^1_{i,n-k}(x) = \emptyset$, and $|H^1_{n-k}(x)| = n-k$ (cf. Eq. (12)). In this case, we have $j = *$. Let $z = x \oplus (2^{n-k} - 1)$ be the node adjacent to x in T_i. Since $i \in H^1_{n,n-k}(x)$, it implies $z_i = x_i = 1$, and thus $i \in H^1_{n,n-k}(z)$. Moreover, $H^1_{i,n-k}(z) = H^1_{i,n-k}(x) = \emptyset$, and $|H^1_{n-k}(x)| = n-k$ implies $H^1_{n-k}(z) = \emptyset$. Thus, $H^1_i(z) = H^1_{i,n-k}(z) \cup H^1_{n-k}(z) = \emptyset$. This shows that z is in the situation of Case 1. Let P be the path connecting z and 0. Therefore, we can find the unique path $T_i[x, 0]$ by concatenating $x \xrightarrow{*} z$ and P.

Case 4: $i \in H^1_{n,n-k}(x)$, $H^1_{i,n-k}(x) = \emptyset$, and $(n-k)/2+\lambda < |H^1_{n-k}(x)| < n-k$ (cf. Eq. (11)). In this case, since $|H^1_{n-k}(x)| < n-k$, it implies $H^0_{n-k}(x) \neq \emptyset$. By Eq. (11), we suppose $j = \max H^0_{n-k}(x) = g_q$ for some $0 \leqslant q \leqslant t-1$. Since $x_j = 0$, x is adjacent to $x+2^j$ in T_i. Let $y = x+2^j$. Clearly $H^0_{n-k}(y) = \{g_{q-1}, g_{q-2}, \ldots, g_0\}$. Moreover, $H^1_{i,n-k}(y) = H^1_{i,n-k}(x) = \emptyset$ and $(n-k)/2 + \lambda < |H^1_{n-k}(x)| < |H^1_{n-k}(y)| \leqslant n-k$. If $|H^1_{n-k}(y)| \neq n-k$, then y is still in the situation of Case 4. By the same argument, we can repeat this process until the path passes through the node $w = x + \sum_{h=0}^{q} 2^{g_h}$. Let Q be the path described as follows: $Q : x \xrightarrow{+2^{g_q}} (x+2^{g_q}) \xrightarrow{+2^{g_{q-1}}} (x+2^{g_q}+2^{g_{q-1}}) \xrightarrow{+2^{g_{q-2}}} \cdots \xrightarrow{+2^{g_0}} w$. Now, it is easy to check that $H^0_n(w) = \{g_{t-1}, g_{t-2}, \ldots, g_{q+1}\}$. Thus, $H^0_{n-k}(w) = \emptyset$ and $|H^1_{n-k}(w)| = n-k$. Moreover, $i \in H^1_{n,n-k}(w)$ and $H^1_{i,n-k}(w) = H^1_{i,n-k}(x) = \emptyset$. Thus, w is in the situation of Case 3. Let P be the path connecting w and 0. Therefore, we can find the unique path $T_i[x, 0]$ by concatenating Q and P.

Case 5: $i \in H^1_{n,n-k}(x)$ and $H^1_{i,n-k}(x) \neq \emptyset$ (cf. Eq. (9)). In this case, since $x_i = 1$ and $H^1_{i,n-k}(x) \neq \emptyset$, by Eq. (9) we suppose $j = \max H^1_{i,n-k}(x) = f_p$ for some $0 \leqslant p < s-1$. A proof similar to that of Case 2 in Claim 1.1 shows that there is a path Q connecting x and a node $w = x - \sum_{h=r}^{p} 2^{f_h}$ such that $H^1_{i,n-k}(w) = \emptyset$, where $0 \leqslant r \leqslant p$. If $r = 0$, then $H^1_i(w) = \emptyset$, and thus w is in the situation of Case 1. Otherwise, $H^1_{n-k}(x) \neq \emptyset$, we check the range of $H^1_{n-k}(x)$ as follows: If $0 < |H^1_{n-k}(w)| \leqslant (n-k)/2+\lambda$, then w is in the situation of Case 2; If $(n-k)/2+\lambda < |H^1_{n-k}(w)| < n-k$, then w is in the situation of Case 4; If $|H^1_{n-k}(w)| = n-k$, then w is in the situation of Case 3. No matter what the situation does w have, let P be the path connecting w and 0. Therefore, we can find the unique path $T_i[x, 0]$ by concatenating Q and P.

Case 6: $i \in H^0_{n,n-k}(x)$ (cf. Eq. (7)). In this case, we have $j = i$. Since $x_j = 0$, x is adjacent to $x + 2^j$ in T_i. Let $z = x + 2^j$. Clearly, $i \in H^1_{n,n-k}(z)$. Thus, z is possible in the situation of any above-mentioned case. Let P be the path that connects z and 0. Thus, we obtain the unique path $T_i[x, 0]$ by concatenating $x \xrightarrow{+2^j} z$ and P.

As a result, this completes the proof. □

Claim 1.3. *If x matches one of the conditions in Eqs.* (13), (14), (15), (16), (17), (18), *then there is a unique path that connects x and 0 in T_i.*

Proof. Let $j = \text{NEXT}_x(i)$ and suppose $x \neq 0$. There are the following scenarios:

Case 1: $i = *$, $H^1_{n,n-k}(x) = \emptyset$, and $H^0_{n-k}(x) = \emptyset$ (cf. Eq. (18)). In this case, we have $j = *$. Let $z = x \oplus (2^{n-k} - 1)$ be the node adjacent to x in T_i. Since $H^1_{n,n-k}(x) = \emptyset$ and $H^0_{n-k}(x) = \emptyset$, it implies $z = 0$. In this case, $x \xrightarrow{*} 0$ is the desired path.

Case 2: $i = *$, $H^1_{n,n-k}(x) = \emptyset$, and $0 < |H^0_{n-k}(x)| \leqslant (n-k)/2$ (cf. Eq. (17)). Recall that we regard '$*$' as the smallest element in $H^0_{n-k}(x) \cup \{*\}$. In this case, since $H^0_{n-k}(x) \neq \emptyset$, by Eq. (17), we suppose $j = \max H^0_{n-k}(x) = g_q$ for some $0 \leqslant q \leqslant t-1$. A proof similar to that of Case (4) in Claim 1.2 shows that there is a path Q connecting x and a node $w = 2^{n-k} - 1$ such that $H^0_{n-k}(w) = \emptyset$. Now, w is in the situation of Case 1. Therefore, we can find the unique path $T_i[x, 0]$

by concatenating Q and $w \xrightarrow{*} 0$.

Case 3: $i = *$, $H^1_{n,n-k}(x) = \emptyset$, and $|H^0_{n-k}(x)| > (n-k)/2$ (cf. Eq. (16)). In this case, we have $j = *$. Let $z = x \oplus (2^{n-k}-1)$ be the node adjacent to x in T_i. Since $x \neq 0$ and $H^1_{n,n-k}(x) = \emptyset$, it implies $|H^1_{n-k}(x)| > 0$ (i.e., $|H^0_{n-k}(x)| < n-k$). Also, $n-k > |H^0_{n-k}(x)| > (n-k)/2$ implies $0 < |H^0_{n-k}(z)| < (n-k)/2$. Moreover, $H^1_{n,n-k}(z) = H^1_{n,n-k}(x) = \emptyset$. Thus, z is in the situation of Case 2. Let P be the path connecting z and 0. Therefore, we obtain the unique path $T_i[x,0]$ by concatenating $x \xrightarrow{*} z$ and P.

Case 4: $i = *$, $H^1_{n,n-k}(x) \neq \emptyset$, and $|H^0_{n-k}(x)| < (n-k)/2 - \lambda$ (cf. Eq. (15)). In this case, since $|H^0_{n-k}(x)| < (n-k)/2-\lambda$, it implies $|H^1_{n-k}(x)| > (n-k)/2+\lambda \geqslant 1.5$. Note that if $n-k = 2$, then $|H^1_2(x)| = 2$, and thus $x_0 = x_1 = 1$. Suppose $H^1_{n,n-k}(x) = \{f_{s-1}, f_{s-2}, \ldots, f_p\}$ for some $2 \leqslant p \leqslant s-1$. By Eq. (15), we have $j = \max H^1_n(x) = f_{s-1}$. Since $x_j = 1$, x is adjacent to $x - 2^j$ in T_i. Let $y = x - 2^j$. Clearly, $H^1_{n,n-k}(y) = \{f_{s-2}, f_{s-3}, \ldots, f_p\}$ and $|H^0_{n-k}(y)| = |H^0_{n-k}(x)| < (n-k)/2 - \lambda$. If $H^1_{n,n-k}(y) \neq \emptyset$, y is still in the situation of Case 4. By the same argument, we can repeat this process until the path passes through the node $w = x - \sum_{h=p}^{s-1} 2^{f_h}$. Let P be the path described as follows: $P : x \xrightarrow{-2^{f_{s-1}}} (x - 2^{f_{s-1}}) \xrightarrow{-2^{f_{s-2}}} (x - 2^{f_{s-1}} - 2^{f_{s-2}}) \xrightarrow{-2^{f_{s-3}}} \cdots \xrightarrow{-2^{f_p}} w$. Now, it is easy to check $H^1_{n,n-k}(w) = \emptyset$ and $|H^0_{n-k}(w)| = |H^0_{n-k}(x)| < (n-k)/2 - \lambda$. Thus, if $H^0_{n-k}(w) = \emptyset$, we have $w = 2^{n-k} - 1$ and it is in the situation of Case 1. Therefore, we obtain the desired path by concatenating P and $w \xrightarrow{*} 0$. Otherwise, w is in the situation of Case 2. Let Q be the path connecting w and 0. Therefore, we can find the unique path $T_i[x,0]$ by concatenating P and Q.

Case 5: $i = *$, $H^1_{n,n-k}(x) \neq \emptyset$, and $(n-k)/2 - \lambda \leqslant |H^0_{n-k}(x)| \leqslant (n-k)/2$ (cf. Eq. (14)). In this case, since $|H^0_{n-k}(x)| \geqslant (n-k)/2-\lambda \geqslant 0.5$, we have $H^0_{n-k}(x) \neq \emptyset$. Note that if $n-k = 2$, then $0.5 = 1 - \lambda \leqslant |H^0_2(x)| \leqslant 1$, and thus $x_0 \oplus x_1 = 1$. By Eq. (14), we suppose $j = \max H^0_{n-k}(x) = g_q$ for some $0 \leqslant q \leqslant t-1$. A proof similar to that of Case (4) in Claim 1.2 shows that there is a path Q connecting x and a node w such that $|H^0_{n-k}(w)| < (n-k)/2 - \lambda$. Now, w is in the situation of Case 4. Let P be the path connecting w and 0. Therefore, we can find the unique path $T_i[x,0]$ by concatenating Q and P.

Case 6: $i = *$, $H^1_{n,n-k}(x) \neq \emptyset$, and $|H^0_{n-k}(x)| > (n-k)/2$ (cf. Eq. (13)). In this case, we have $j = *$. Let $z = x \oplus (2^{n-k} - 1)$ be the node adjacent to x in T_i. Since $|H^0_{n-k}(x)| > (n-k)/2$, it implies $|H^0_{n-k}(z)| < (n-k)/2$. Since $H^1_{n,n-k}(z) = H^1_{n,n-k}(x) \neq \emptyset$, z is in the situation of Case 4 or Case 5. No matter what the situation does z have, let P be the path connecting z and 0. Therefore, we obtain the unique path $T_i[x,0]$ by concatenating $x \xrightarrow{*} z$ and P.

As a result, this completes the proof. ☐

Appendix B
Proofs of Claims in Theorem 2.

The following lemmas shows the independency of spanning trees. For convenience, if P is a path and $u, v \in V(P)$, we use $P(u,v)$ to denote the subpath of P from u to v. Also, we write $P(u,v)_i = b$, where $0 \leqslant i \leqslant n-1$ and $b \in \{0,1\}$, to mean that $x_i = b$ for every node $x = x_{n-1}x_{n-2}\cdots x_0$ in $P(u,v)$.

Claim 2.1. *If $i, j \in H^1_{n-k}$, then $P || Q$.*

Proof. Without loss of generality, we suppose $n-k > i > j \geqslant 0$. Note that $x_i = x_j = 1$. There are three cases as follows.

Case 1: $|H^1_{n-k}(x)| \leqslant (n-k)/2 + \lambda$. From the paths constructed in Claim 1.1 (cf. Eqs. (4) and (5)), we know that P starts with an edge labeled by $-2^{i'}$, where $i' = \textsc{next}_x(i)$, and ends with an edge labeled by -2^i. Since $x_{i'} = 1$ and it has been changed to 0 when P passes through the first edge, we have $P(x - 2^{i'}, 2^i)_{i'} = 0$. Also, since x_i remains unchanged until P passes through the last edge, we have $P(x - 2^{i'}, 2^i)_i = 1$. Similarly, Q starts with an edge labeled by $-2^{j'}$, where $j' = \textsc{next}_x(j)$, and ends with an edge labeled by -2^j. Clearly, the path $Q(x - 2^{j'}, 2^j)$ contains an edge with label -2^i, denoted by $w \xrightarrow{-2^i} w'$. Since $x_{i'}$ alters after the change of x_i in Q, we have $Q(x - 2^{j'}, w)_{i'} = 1$ and $Q(w', 2^j)_i = 0$. As a result, every node of $P(x - 2^{i'}, 2^i)$ has a bit different from nodes of $Q(x - 2^{j'}, w) \cup Q(w', 2^j)$.

Case 2: $|H^1_{n-k}(x)| > (n-k)/2 + \lambda$ (for $n-k > 2$). In this case, we have $\textsc{next}_x(i) = i$ and $\textsc{next}_x(j) = j$. Thus, P (respectively, Q) has the label -2^i (respectively, -2^j) in its first edge and last edge. Since $n-k > 2$, we have $|H^1_{n-k}(x)| \geqslant 3$, and there is a position $\ell \in H^1_{n-k}(x) \setminus \{i,j\}$ such that $x_\ell = 1$. From the paths constructed in Claim 1.1 (cf. Eq. (6)), we know that P and Q must have passed through an edge with label $*$, denoted by $w_P \xrightarrow{*} w'_P$ and $w_Q \xrightarrow{*} w'_Q$, respectively. Since x_i has been changed to 0 when P passes through the first edge, we have $P(x-2^i, w_P)_i = 0$ and $P(w'_P, 2^i)_i = 1$. Moreover, since $P(x-2^i, w_P)$ never changes a bit from 1 to 0 after the change of x_i, it follows that $P(x-2^i, w_P)_\ell = 1$ and $P(w'_P, 2^i)_\ell = 0$. On the other hand, since $Q(x-2^j, w_Q)$ does not contain an edge with label -2^i, we have $Q(x-2^j, w_Q)_i = 1$ and $Q(w'_Q, 2^j)_i = 0$. Again, since $Q(x-2^j, w_Q)$ never changes a bit from 1 to 0 after the change of x_j, it follows that $Q(x-2^j, w_Q)_\ell = 1$ and $Q(w'_Q, 2^j)_\ell = 0$. As a result, every node of $P(x-2^i, w_P) \cup P(w'_P, 2^i)$ has a bit different from nodes of $Q(x-2^j, w_Q) \cup Q(w'_Q, 2^j)$.

Case 3: $|H^1_2(x)| = 2$ (for $n-k = 2$). From the paths constructed in Claim 1.1 (cf. Eq. (6′)), we have $|H^1_2(x)| = 2$. Since $i > j$, we have $i = 1$ and $j = 0$. Let $i' = \max H^1_{2-i}(x) = 0$ and $j' = \max H^1_{2-j}(x) = 1$. The proof is similar to Case 1. ☐

Claim 2.2. *If $i, j \in H^0_{n-k}$, then $P||Q$.*

Proof. Without loss of generality, we suppose $n-k > i > j \geqslant 0$. Note that $x_i = x_j = 0$. There are two cases as follows.

Case 1: $|H^0_{n-k}(x)| \leqslant (n-k)/2$. From the paths constructed in Claim 1.1 (cf. Eqs. (1) and (2)), P starts with an edge labeled by $+2^{i'}$, where $i' = \text{NEXT}_x(i)$, and ends with an edge labeled by -2^i. Similarly, Q starts with an edge labeled by $+2^{j'}$, where $j' = \text{NEXT}_x(j)$, and ends with an edge labeled by -2^j. Moreover, P and Q can be described as follows: $P: x \xrightarrow{+2^{i'}} (x+2^{i'}) \to \cdots \to u \xrightarrow{+2^j} u' \xrightarrow{+2^{j'}} u'' \to \cdots \to w_P \xrightarrow{*} w'_P \to \cdots \to (2^i) \xrightarrow{-2^i} 0$ and $Q: x \xrightarrow{+2^{j'}} (x+2^{j'}) \to \cdots \to w_Q \xrightarrow{*} w'_Q \to \cdots \to v \xrightarrow{-2^i} v' \xrightarrow{-2^{i'}} v'' \to \cdots \to (2^j) \xrightarrow{-2^j} 0$. Note that $x_{i'} = 0$ and it is possible $i' = j$ or $j' = *$ (we ignore the relevant subpaths in this case). If $j' \neq *$, then $x_{j'} = 0$. By carefully analyzing the alteration of bits, the bits in positions i, i', j and j' for nodes in P are as follows: $P(x+2^{i'}, u)_i = P(u', u')_i = P(u'', w_P)_i = P(w'_P, 2^i)_{i'} = P(x+2^{i'}, u)_j = P(w'_P, 2^i)_j = P(x+2^{i'}, u)_{j'} = P(u', u')_{j'} = P(w'_P, 2^i)_{j'} = 0$ and $P(w'_P, 2^i)_i = P(x+2^{i'}, u)_{i'} = P(u', u')_{i'} = P(u'', w_P)_{i'} = P(u', u')_j = P(u'', w_P)_j = P(u'', w_P)_{j'} = 1$.

Similarly, the bits in positions i, i', j and j' for nodes in Q are as follows: $Q(x+2^{j'}, w_Q)_i = Q(v', v')_i = Q(v'', 2^j)_i = Q(x+2^{j'}, w_Q)_{i'} = Q(x+2^{j'}, w_Q)_j = Q(w'_Q, v)_{j'} = Q(v', v')_{j'} = Q(v'', 2^j)_{j'} = 0$ and $Q(w'_Q, v)_i = Q(w'_Q, v)_{i'} = Q(v', v')_{i'} = Q(v'', 2^j)_{i'} = Q(w'_Q, v)_j = Q(v', v')_j = Q(v'', 2^j)_j = Q(x+2^{j'}, w_Q)_{j'} = 1$.

We observe that only $P(u', u')$ and $Q(v', v')$ have the same setting in these bits. Since $|H^0_{n-k}(x)| \leqslant (n-k)/2$, it implies $|H^1_{n-k}(x)| \geqslant (n-k)/2$, and thus there is a position $\ell \in H^1_{n-k}(x)$ such that $x_\ell = 1$. Since x_ℓ remains unchanged until P passes through the edge with label $*$, we have $P(u', u')_\ell = 1$. By contrast, x_ℓ has been changed to 0 when Q passes through the edge with label $*$, we have $Q(v', v')_\ell = 0$. This shows that $P(x+2^{i'}, 2^i) \cap Q(x+2^{j'}, 2^j) = \emptyset$.

Case 2: $|H^0_{n-k}(x)| > (n-k)/2$. From the paths constructed in Claim 1.1 (cf. Eq. (3)), we have $\text{NEXT}_x(i) = i$ and $\text{NEXT}_x(j) = j$. Thus, P (respectively, Q) has the label $+2^i$ (respectively, $+2^j$) in its first edge and the label -2^i (respectively, -2^j) in its last edge. Since x_i has been changed to 1 when P passes through the first edge, we have $P(x+2^i, 2^i)_i = 1$. On the other hand, since $Q(x+2^j, 2^j)$ never changes a bit from 0 to 1 after the change of x_j, we have $Q(x+2^j, 2^j)_i = 0$. Thus, $P(x+2^i, 2^i) \cap Q(x+2^j, 2^j) = \emptyset$. □

Claim 2.3. *If $i, j \in H^1_{n,n-k}$, then $P||Q$.*

Proof. Without loss of generality, we suppose $n > i > j \geqslant n-k$. Since $i, j \in H^1_{n,n-k}(x)$, we have $H^1_{i,n-k}(x) \neq \emptyset$. From the paths constructed in Claim 1.2 (cf. Eq. (9)), we know that P starts with an edge labeled by $-2^{i'}$, where $i' = \max H^1_i(x)$, and ends with an edge labeled by -2^i. Note that it is possible $i' = j$. Since $x_{i'} = 1$ and it has been changed to 0 when P passes through the first edge, we have $P(x-2^{i'}, 2^i)_{i'} = 0$. Also, since x_i remains unchanged until P passes through the last edge, we have $P(x-2^{i'}, 2^i)_i = 1$. There are two cases as follows.

Case 1: $H^1_{j,n-k}(x) \neq \emptyset$ or $|H^1_{n-k}(x)| \leqslant (n-k)/2 + \lambda$. From the paths constructed in Claim 1.2 (cf. Eqs. (8), (9) and (10)), Q starts with an edge labeled by $-2^{j'}$, where $j' = \text{NEXT}_x(j)$, and ends with an edge labeled by -2^j. Clearly, the path $Q(x-2^{j'}, 2^j)$ contains an edge with label -2^i, denoted by $w \xrightarrow{-2^i} w'$. Since $x_{i'}$ alters after the change of x_i in Q, we have $Q(x-2^{j'}, w)_{i'} = 1$ and $Q(w', 2^j)_i = 0$. As a result, every node of $P(x-2^{i'}, 2^i)$ has a bit different from nodes of $Q(x-2^{j'}, w) \cup Q(w', 2^j)$.

Case 2: $H^1_{j,n-k}(x) = \emptyset$ and $(n-k)/2 + \lambda < |H^1_{n-k}(x)| \leqslant n-k$. Let $j' = \text{NEXT}_x(j)$. From the paths constructed in Claim 1.2 (cf. Eqs. (11) and (12)), Q starts with an edge labeled by $+2^{j'}$ or $*$, and ends with an edge labeled by -2^j. If $j' \neq *$, then $x_{j'} = 0$. It follows that Q must have passed through an edge labeled by $*$. Since Q contains an edge with label -2^i, denoted by $w \xrightarrow{-2^i} w'$, an argument similar to Case 1 shows that $Q(x+2^{j'}, w)_{i'} = 1$ and $Q(w', 2^j)_i = 0$. Thus, there is a different bit between nodes of $P(x-2^{i'}, 2^i)$ and $Q(x+2^{j'}, w) \cup Q(w', 2^j)$. □

Claim 2.4. *If $i, j \in H^0_{n,n-k}$, then $P||Q$.*

Proof. Note that $x_i = x_j = 0$. From the paths constructed in Claim 1.2 (cf. Eq. (7)), we have $\text{NEXT}_x(i) = i$ and $\text{NEXT}_x(j) = j$. Thus, P (respectively, Q) has the label $+2^i$ (respectively, $+2^j$) in its first edge and the label -2^i (respectively, -2^j) in its last edge. Since x_i (respectively, x_j) has been changed to 1 when P (respectively, Q) passes through the first edge, we have $P(x+2^i, 2^i)_i = Q(x+2^j, 2^j)_j = 1$. Also, since P (respectively, Q) does not contain an edge with label $+2^j$ (respectively, $+2^i$), we have $P(x+2^i, 2^i)_j = Q(x+2^j, 2^j)_i = 0$. Thus, $P(x+2^i, 2^i) \cap Q(x+2^j, 2^j) = \emptyset$. □

Claim 2.5. *If $i \in H^1_{n-k}$ and $j \in H^0_{n-k}$, then $P||Q$.*

Proof. Note that $x_i = 1$ and $x_j = 0$. There are four cases as follows.

Case 1: $|H^1_{n-k}(x)| < (n-k)/2$. This implies $|H^0_{n-k}(x)| > (n-k)/2$. From the paths constructed in Claim 1.1 (cf. Eqs. (4) and (5)), P starts with an edge labeled by $-2^{i'}$, where $i' = \text{NEXT}_x(i)$, and ends with an edge labeled by -2^i. By contrast, from the paths constructed in Claim 1.1 (cf. Eq. (3)), Q starts with an edge labeled by $+2^j$ because $\text{NEXT}_x(j) = j$, and ends with an edge labeled by -2^j. Since $P(x-2^{i'}, 2^i)$ never changes a bit from 0 to 1, we have $P(x-2^{i'}, 2^i)_j = 0$.

On the other hand, since x_j has been changed to 1 when Q passes through the first edge and then keeps unchanged until Q passes through the last edge, we have $Q(x+2^j,2^j)_j = 1$. Thus, $P(x-2^{i'},2^i) \cap Q(x+2^j,2^j) = \emptyset$.

Case 2: $(n-k)/2 \leqslant |H^1_{n-k}(x)| \leqslant (n-k)/2+\lambda$. This implies $(n-k)/2-\lambda \leqslant |H^0_{n-k}(x)| \leqslant (n-k)/2$. In this case, P is the same as that described in Case 1. Let $j' = \text{NEXT}_x(j)$. From the paths constructed in Claim 1.1 (cf. Eqs. (1) and (2)), Q starts with an edge labeled by $+2^{j'}$ or $*$, and ends with an edge labeled by -2^j. If $j' \neq *$, then $x_{j'} = 0$. It follows that Q must have passed through an edge with label $*$, denoted by $w_Q \xrightarrow{*} w'_Q$. Since $P(x-2^{i'},2^i)$ never changes a bit from 0 to 1, we have $P(x-2^{i'},2^i)_j = 0$ and $P(x-2^{i'},2^i)_{j'} = 0$ for $j' \neq *$. On the other hand, since $x_{j'}$ has been changed to 1 when Q passes through the first edge, we have $Q(x+2^{j'},w_Q)_{j'} = 1$ and $Q(w'_Q,2^j)_{j'} = 0$. Moreover, since $Q(x+2^{j'},w_Q)$ does not contain an edge with label $+2^j$, we have $Q(x+2^{j'},w_Q)_j = 0$ and $Q(w'_Q,2^j)_j = 1$. Thus, there is a different bit between nodes of $P(x-2^{i'},2^i)$ and $Q(x+2^{j'},w_Q) \cup Q(w'_Q,2^j)$.

Case 3: $|H^1_{n-k}(x)| > (n-k)/2+\lambda$ (for $n-k > 2$). This implies $|H^0_{n-k}(x)| < (n-k)/2-\lambda$. Since $n-k > 2$, we have $|H^1_{n-k}(x)| \geqslant 3$ and there is a position $\ell \in H^1_{n-k}(x) \setminus \{i\}$ such that $x_\ell = 1$. From the paths constructed in Claim 1.1 (cf. Eq. (6)), we have $\text{NEXT}_x(i) = i$ and P has the label -2^i in its first edge and last edge. In this case, Q is the same as that described in Case 2. Note that both P and Q must have passed through an edge with label $*$, denoted by $w_P \xrightarrow{*} w'_P$ and $w_Q \xrightarrow{*} w'_Q$ respectively. Since x_i has been changed to 0 when P passes through the first edge, we have $P(x-2^i,w_P)_i = 0$ and $P(w'_P,2^i)_i = 1$. Also, since $P(x-2^i,w_P)$ never changes a bit from 1 to 0 after the change of x_i, it follows that $P(x-2^i,w_P)_\ell = 1$ and $P(w'_P,2^i)_\ell = 0$. On the other hand, since $Q(x+2^{j'},w_Q)$ never changes a bit from 1 to 0, we have $Q(x+2^{j'},w_Q)_i = Q(x+2^{j'},w_Q)_\ell = 1$ and $Q(w'_Q,2^j)_i = Q(w'_Q,2^j)_\ell = 0$. Thus, there is a different bit between nodes of $P(x-2^i,w_P) \cup P(w'_P,2^i)$ and $Q(x+2^{j'},w_Q) \cup Q(w'_Q,2^j)$.

Case 4: $|H^1_2(x)| = 2$ (for $n-k = 2$). This case is impossible because $|H^1_{n-k}(x)| = 2$ and $j \in H^0_{n-k}(x)$. ☐

Claim 2.6. *If $i \in H^1_{n-k}$ and $j \in H^1_{n,n-k'}$, then $P||Q$.*

Proof. Note that $x_i = x_j = 1$. There are two cases as follows.

Case 1: $H^1_{j,n-k}(x) \neq \emptyset$ or $|H^1_{n-k}(x)| \leqslant (n-k)/2+\lambda$. From the paths constructed in Claim 1.1 (cf. Eqs. (4) and (5)), P starts with an edge labeled by $-2^{i'}$, where $i' = \text{NEXT}_x(i)$, and ends with an edge labeled by -2^i. By contrast, from the paths constructed in Claim 1.2 (cf. Eqs. (8), (9) and (10)), Q starts with an edge labeled by $-2^{j'}$, where $j' = \text{NEXT}_x(j)$, and ends with an edge labeled by -2^j. Note that it is possible $i' = j$ or $j' = i$. If $j' \neq i$, then $Q(x-2^{j'},2^j)$ contains an edge with label -2^i, denoted by $w \xrightarrow{-2^i} w'$. Since $x_{i'}$ alters after the change of x_i in Q, we have $Q(x-2^{j'},w)_{i'} = 1$ and $Q(w',2^j)_i = 0$. On the other hand, since $x_{i'}$ has been changed to 0 when P passes through the first edge, we have $P(x-2^{i'},2^i)_{i'} = 0$. Also, since x_i remains unchanged until P passes through the last edge, we have $P(x-2^{i'},2^i)_i = 1$. This shows that every node of $P(x-2^{i'},2^i)$ has a bit different from nodes of $Q(x-2^{j'},w) \cup Q(w',2^j)$.

Case 2: $H^1_{j,n-k}(x) = \emptyset$ and $(n-k)/2+\lambda < |H^1_{n-k}(x)| \leqslant n-k$. Since $|H^1_{n-k}(x)| > (n-k)/2+\lambda$, there is a position $\ell \in H^1_{n-k}(x) \setminus \{i\}$ such that $x_\ell = 1$. Let $j' = \text{NEXT}_x(j)$. From the paths constructed in Claim 1.2 (cf. Eqs. (11) and (12)), Q starts with an edge labeled by $+2^{j'}$ or $*$, and ends with an edge labeled by -2^j. If $j' \neq *$, then $j' = \max H^0_{n-k}(x)$ and Q must have passed through an edge with label $*$, denoted by $w_Q \xrightarrow{*} w'_Q$. For $n-k > 2$, from the paths constructed in Claim 1.1 (cf. Eq. (6)), we have $\text{NEXT}_x(i) = i$ and P has the label -2^i in its first edge and last edge. Note that P must have passed through an edge with label $*$, denoted by $w_P \xrightarrow{*} w'_P$. Since x_i has been changed to 0 when P passes through the first edge, we have $P(x-2^i,w_P)_i = 0$ and $P(w'_P,2^i)_i = 1$. Also, since $P(x-2^i,w_P)$ never changes a bit from 1 to 0 except the first edge, we have $P(x-2^i,w_P)_\ell = 1$ and $P(w'_P,2^i)_\ell = 0$. On the other hand, since $Q(x+2^{j'},w_Q)$ never changes a bit from 1 to 0, we have $Q(x+2^{j'},w_Q)_i = Q(x+2^{j'},w_Q)_\ell = 1$ and $Q(w'_Q,2^j)_i = Q(w'_Q,2^j)_\ell = 0$. This shows that there is a different bit between nodes of $P(x-2^i,w_P) \cup P(w'_P,2^i)$ and $Q(x+2^{j'},w_Q) \cup Q(w'_Q,2^j)$.

For $n-k = 2$, from the paths constructed in Claim 1.1 (cf. Eq. (6′)), P starts with an edge labeled by $-2^{i'}$, where $i' = \max H^1_{2-i}(x)$, and ends with an edge labeled by -2^i. Since $x_{i'}$ has been changed to 0 when P passes through the first edge, we have $P(x-2^{i'},2^i)_{i'} = 0$. Also, since x_i remains unchanged until P passes through the last edge, we have $P(x-2^{i'},2^i)_i = 1$. On the other hand, since $Q(x+2^{j'},w_Q)$ never changes a bit from 1 to 0, we have $Q(x+2^{j'},w_Q)_i = Q(x+2^{j'},w_Q)_{i'} = 1$ and $Q(w'_Q,2^j)_i = Q(w'_Q,2^j)_{i'} = 0$. This shows that there is a different bit between nodes of $P(x-2^{i'},2^i)$ and $Q(x+2^{j'},w_Q) \cup Q(w'_Q,2^j)$. ☐

Claim 2.7. *If $i \in H^1_{n-k}$ and $j \in H^0_{n,n-k'}$, then $P||Q$.*

Proof. Note that $x_i = 1$ and $x_j = 0$. Let w be the node adjacent to x in P. From the paths constructed in Claim 1.1 (cf. Eqs. (4), (5), (6) and (6′)), we know that P ends with an edge labeled by -2^i and never changes a bit in $H^0_{n,n-k}(x)$ from 0 to 1. Since $j \in H^0_{n,n-k'}$, we have $P(w,2^i)_j = 0$. On the other hand, from the paths constructed in Claim 1.2 (cf. Eq. (7)), we have $\text{NEXT}_x(j) = j$. Thus, Q has the label $+2^j$ in its first edge and the label -2^j in its last edge. Since x_j has been changed to 1 when Q passes through the first edge and then keeps unchanged until Q passes through the last edge, we have $Q(x+2^j, 2^j)_j = 1$. Thus $P(w, 2^i) \cap Q(x+2^j, 2^j) = \emptyset$. □

Claim 2.8. *If $i \in H^0_{n-k}$ and $j \in H^1_{n,n-k}$, then $P||Q$.*

Proof. Note that $x_i = 0$ and $x_j = 1$. There are two cases as follows.

Case 1: $H^1_{j,n-k}(x) \neq \emptyset$ or $|H^1_{n-k}(x)| \leqslant (n-k)/2+\lambda$. This implies $|H^0_{n-k}(x)| \geqslant (n-k)/2-\lambda$. In this case, Q is the same as that in Case 1 of Claim 2.6. We first consider $(n-k)/2 \geqslant |H^0_{n-k}(x)| \geqslant (n-k)/2-\lambda$. Let $i' = \text{NEXT}_x(i)$. From the paths constructed in Claim 1.1 (cf. Eqs. (1) and (2)), P starts with an edge labeled by $+2^{i'}$ or $*$, and ends with an edge labeled by -2^i. If $i' \neq *$, then $x_{i'} = 0$ and P must have passed through an edge with label $*$, denoted by $w \xrightarrow{*} w'$. Since $x_{i'}$ has been changed to 1 when P passes through the first edge, we have $P(x+2^{i'}, w)_{i'} = 1$ and $P(w', 2^i)_{i'} = 0$. Also, since x_i remains unchanged until P passes through the edge $w \xrightarrow{*} w'$, we have $P(x-2^i, w)_i = 0$ and $P(w', 2^i)_i = 1$. On the other hand, since Q never changes a bit from 0 to 1, we have $Q(x-2^{j'}, 2^j)_i = Q(x-2^{j'}, 2^j)_{i'} = 0$. Thus, every node of $P(x+2^i, w) \cup P(w', 2^i)$ has a bit different from nodes of $Q(x-2^{j'}, 2^j)$.

Next, we consider $|H^0_{n-k}(x)| > (n-k)/2$. From the paths constructed in Claim 1.1 (cf. Eq. (3)), we have $\text{NEXT}_x(i) = i$. Thus, P has the label $+2^i$ in its first edge and the label -2^i in its last edge. Since x_i has been changed to 1 when P passes through the first edge and then keeps unchanged until P passes through the last edge, we have $P(x+2^i, 2^i)_i = 1$. On the other hand, since Q never changes a bit from 0 to 1, we have $Q(x-2^{j'}, 2^j)_i = 0$. Thus $P(x+2^i, 2^i) \cap Q(x-2^{j'}, 2^j) = \emptyset$.

Case 2: $H^1_{j,n-k}(x) = \emptyset$ and $(n-k)/2+\lambda < |H^1_{n-k}(x)| \leqslant n-k$. Since $|H^1_{n-k}(x)| > (n-k)/2+\lambda$, it implies that $|H^0_{n-k}(x)| < (n-k)/2-\lambda$ and there is a position $\ell \in H^1_{n-k}(x)$ such that $x_\ell = 1$. Let $i' = \text{NEXT}_x(i)$. From the paths constructed in Claim 1.1 (cf. Eqs. (1) and (2)), P starts with an edge labeled by $+2^{i'}$ or $*$, and ends with an edge labeled by -2^i. If $i' \neq *$, then $x_{i'} = 0$ and P must have passed through an edge with label $*$, denoted by $w_P \xrightarrow{*} w'_P$. Since $|H^0_{n-k}(x)| \leqslant (n-k)/2$, the path Q is the same as that in Case 2 of Claim 2.6. Let $j' = \text{NEXT}_x(j)$. That is, Q starts with an edge labeled by $+2^{j'}$ or $*$, and ends with an edge labeled by -2^j. If $j' \neq *$, then $j' = \max H^0_{n-k}(x) \geqslant i$ and Q must have passed through an edge with label $*$, denoted by $w_Q \xrightarrow{*} w'_Q$. Furthermore, if $j' > i$, the path $P(w'_P, 2^i)$ contains an edge with label $-2^{j'}$, denoted by $u \xrightarrow{-2^{j'}} u'$, and the path $Q(x+2^{j'}, w_Q)$ contains an edge with label $+2^i$, denoted by $v \xrightarrow{+2^i} v'$. It is clear that the bits in positions i, j' and ℓ for nodes in P are as follows: $P(x+2^{i'}, w_P)_i = P(x+2^{i'}, w_P)_{j'} = P(u', 2^i)_{j'} = P(w'_P, u)_\ell = P(u', 2^i)_\ell = 0$ and $P(w'_P, u)_i = P(u', 2^i)_i = P(w'_P, u)_{j'} = P(x+2^{i'}, w_P)_\ell = 1$. On the other hand, the bits in positions i, j' and ℓ for nodes in Q are as follows: $Q(x+2^{j'}, v)_i = Q(w'_Q, 2^j)_i = Q(w'_Q, 2^j)_{j'} = Q(w'_Q, 2^j)_\ell = 0$ and $Q(v', w_Q)_i = Q(x+2^{j'}, v)_{j'} = Q(v', w_Q)_{j'} = Q(x+2^{j'}, v)_\ell = Q(v', w_Q)_\ell = 1$. This show that $P(x+2^{i'}, 2^i) \cap Q(x+2^{j'}, 2^j) = \emptyset$. □

Claim 2.9. *If $i \in H^0_{n-k}$ and $j \in H^0_{n,n-k}$, then $P||Q$.*

Proof. Note that $x_i = x_j = 0$. The proof is the same as that in Claim 2.7, except for the path P constructed in Claim 1.1 for Eqs. (1), (2) and (3). □

Claim 2.10. *If $i \in H^1_{n,n-k}$ and $j \in H^0_{n,n-k}$, then $P||Q$.*

Proof. Note that $x_i = 1$ and $x_j = 0$. Let w be the node adjacent to x in P. From the paths constructed in Claim 1.2 (cf. Eqs. from (8) to (12)), P ends with an edge labeled by -2^i and never changes a bit in $H^0_{n,n-k}(x)$ from 0 to 1. Since $j \in H^0_{n,n-k}$, we have $P(w, 2^i)_j = 0$. In this case, Q is the same as that in Claim 2.7. Thus, we can show $P(w, 2^i) \cap Q(x+2^j, 2^j) = \emptyset$ using a similar argument. □

Claim 2.11. *If $i \in H^1_{n-k}$ and $j = *$, then $P||Q$.*

Proof. Note that $x_i = 1$. There are four cases as follows.

Case 1: $|H^1_{n-k}(x)| < (n-k)/2$. This implies $|H^0_{n-k}(x)| > (n-k)/2$, and thus there is a position $\ell \in H^0_{n-k}(x)$ such that $x_\ell = 0$. From the paths constructed in Claim 1.1 (cf. Eqs. (4) and (5)), P starts with an edge labeled by $-2^{i'}$, where $i' = \text{NEXT}_x(i)$, and ends with an edge labeled by -2^i. Also, from the paths constructed in Claim 1.3 (cf. Eqs. (13) and (16)), Q has the label $*$ in its first edge and last edge. Since $P(x-2^{i'}, 2^i)$ never changes a bit from 0 to 1, we have $P(x-2^{i'}, 2^i)_\ell = 0$. On the other hand, since x_ℓ has been changed to 1 when Q passes through the first edge and then keeps unchanged until Q passes through the last edge, we have $Q(x \oplus (2^{n-k}-1), 2^{n-k}-1)_\ell = 1$. Thus, $P(x-2^i, 2^i) \cap Q(x \oplus (2^{n-k}-1), 2^{n-k}-1) = \emptyset$.

Case 2: $(n-k)/2 \leqslant |H^1_{n-k}(x)| \leqslant (n-k)/2+\lambda$. This implies $(n-k)/2-\lambda \leqslant |H^0_{n-k}(x)| \leqslant (n-k)/2$. In this case, P is the same as that described in Case 1. Also, from the paths constructed in Claim 1.3 (cf. Eqs. (14) and (17)), Q starts with an edge labeled by $+2^{j'}$, where $j' = \max H^0_{n-k}(x)$, and ends with an edge labeled by $*$. Since $P(x-2^{i'}, 2^i)$ never changes a bit from 0 to 1, we have $P(x-2^{i'}, 2^i)_{j'} = 0$. On the other hand, since $x_{j'}$ has been changed to 1 when Q passes through the first edge and then keeps unchanged until Q passes through the last edge, we have $Q(x+2^{j'}, 2^{n-k}-1)_{j'} = 1$. Thus, $P(x-2^i, 2^i) \cap Q(x+2^{j'}, 2^{n-k}-1) = \emptyset$.

Case 3: $H^1_{n,n-k}(x) = \emptyset$ and $|H^1_{n-k}(x)| > (n-k)/2+\lambda$. This implies $|H^0_{n-k}(x)| < (n-k)/2-\lambda$, and thus there is a position $\ell \in H^1_{n-k}(x)\backslash\{i\}$ such that $x_\ell = 1$. Let $j' = \text{NEXT}_x(j)$. From the paths constructed in Claim 1.3 (cf. Eqs. (17) and (18)), Q starts with an edge labeled by $+2^{j'}$ or $*$, and ends with an edge labeled by $*$. For $n-k > 2$, from the paths constructed in Claim 1.1 (cf. Eq. (6)), we have $\text{NEXT}_x(i) = i$ and P has the label -2^i in its first edge and last edge. Note that P must have passed through an edge with label $*$, denoted by $w_P \xrightarrow{*} w'_P$. Since x_i has been changed to 0 when P passes through the first edge, we have $P(x-2^i, w_P)_i = 0$ and $P(w'_P, 2^i)_i = 1$. Moreover, since $P(x-2^i, w_P)$ never changes a bit from 1 to 0 after the change of x_i, it follows that $P(x-2^i, w_P)_\ell = 1$ and $P(w'_P, 2^i)_\ell = 0$. On the other hand, since $Q(x+2^{j'}, 2^{n-k}-1)$ never changes a bit from 1 to 0, we have $Q(x+2^{j'}, 2^{n-k}-1)_i = Q(x+2^{j'}, 2^{n-k}-1)_\ell = 1$. Thus, there is a different bit between nodes of $P(x-2^i, w_P) \cup P(w'_P, 2^i)$ and $Q(x+2^{j'}, 2^{n-k}-1)$.

For $n-k = 2$, from the paths constructed in Claim 1.1 (cf. Eq. (6′)), we have $i \in H^1_2$. Thus, P starts with an edge labeled by $-2^{i'}$, where $i' = \max H^1_{2-i}(x)$, and ends with an edge labeled by -2^i. Since $x_{i'}$ has been changed to 0 when P passes through the first edge, we have $P(x-2^{i'}, 2^i)_{i'} = 0$. On the other hand, since $Q(x+2^{j'}, 2^{n-k}-1)$ never changes a bit from 1 to 0, we have $Q(x+2^{j'}, 2^{n-k}-1)_{i'} = 1$. Thus, $P(x-2^{i'}, 2^i) \cap Q(x+2^{j'}, 2^{n-k}-1) = \emptyset$.

Case 4: $H^1_{n,n-k}(x) \neq \emptyset$ and $|H^1_{n-k}(x)| > (n-k)/2+\lambda$. This implies $|H^0_{n-k}(x)| < (n-k)/2-\lambda$. In this case, P is the same as that described in Case 3. From the paths constructed in Claim 1.3 (cf. Eq. (15)), Q starts with an edge labeled by $-2^{j'}$, where $j' = \max H^1_n(x)$, and ends with an edge labeled by $*$. For $n-k > 2$, since $H^1_{n,n-k} \neq \emptyset$, we have $j' > i$. By the same argument as that in Case 3, we can show that $P(x-2^i, w_P)_i = P(w'_P, 2^i)_\ell = 0$ and $P(w'_P, 2^i)_i = P(x-2^i, w_P)_\ell = 1$. On the other hand, since $Q(x-2^{j'}, 2^{n-k}-1)$ never changes a bit from 1 to 0 in $H^1_{n-k}(x)$, we have $Q(x-2^{j'}, 2^{n-k}-1)_\ell = Q(x-2^{j'}, 2^{n-k}-1)_i = 1$. This shows that every node of $P(x-2^i, w_P) \cup P(w'_P, 2^i)$ has a bit different from nodes of $Q(x-2^{j'}, 2^{n-k}-1)$.

For $n-k = 2$, by the same argument as that in Case 3, we have $P(x-2^{i'}, 2^i)_{i'} = 0$. Since $H^1_{n,n-k} \neq \emptyset$, we have $j' > i'$. On the other hand, since $Q(x-2^{j'}, 2^{n-k}-1)$ does not contain an edge with label $-2^{i'}$, we have $Q(x-2^{j'}, 2^{n-k}-1)_{i'} = 1$. Thus, $P(x-2^{i'}, 2^i) \cap Q(x-2^{j'}, 2^{n-k}-1) = \emptyset$. □

Claim 2.12. *If $i \in H^0_{n-k}$ and $j = *$, then $P||Q$.*

Proof. Note that $x_i = 0$. There are four cases as follows.

Case 1: $|H^0_{n-k}(x)| > (n-k)/2$. Since $|H^0_{n-k}(x)| \geqslant 2$, there is a position $\ell \in H^0_{n-k}(x) \setminus \{i\}$ such that $x_\ell = 0$. In this case, $\text{NEXT}_x(i) = i$. From the paths constructed in Claim 1.1 (cf. Eq. (3)), P has the label $+2^i$ in its first edge and the label -2^i in its last edge. Also, from the paths constructed in Claim 1.3 (cf. Eqs. (13) and (16)), Q has the label $*$ in its first edge and last edge. Since $P(x+2^i, 2^i)$ never changes a bit from 0 to 1 after the change of x_i, we have $P(x+2^i, 2^i)_\ell = 0$. On the other hand, since x_ℓ has been changed to 1 when Q passes through the first edge and then keeps unchanged until Q passes through the last edge, we have $Q(x \oplus (2^{n-k}-1), 2^{n-k}-1)_\ell = 1$. Thus, there is a different bit between nodes of $P(x+2^i, 2^i)$ and $Q(x \oplus (2^{n-k}-1), 2^{n-k}-1)$.

Case 2: $H^1_{n,n-k}(x) \neq \emptyset$ and $(n-k)/2-\lambda \leqslant |H^0_{n-k}(x)| \leqslant (n-k)/2$ or $H^1_{n,n-k}(x) = \emptyset$ and $0 < |H^0_{n-k}(x)| \leqslant (n-k)/2$. This implies that $|H^1_{n-k}(x)| \geqslant (n-k)/2 \geqslant 1$, and thus there is a position $\ell \in H^1_{n-k}(x)$ such that $x_\ell = 1$. Let $i' = \text{NEXT}_x(i)$. From the paths constructed in Claim 1.1 (cf. Eqs. (1) and (2)), P starts with an edge labeled by $+2^{i'}$ or $*$, and ends with an edge labeled by -2^i. If $i' \neq *$, then P must have passed through an edge with label $*$, denoted by $w_P \xrightarrow{*} w'_P$. Also, from the paths constructed in Claim 1.3 (cf. Eqs. (14) and (17)), Q starts with an edge labeled by $+2^{j'}$, where $j' = \max H^0_{n-k}(x)$, and ends with an edge labeled by $*$. Note that $j' \geqslant i$. Since $P(x+2^{i'}, w_P)$ does not contain an edge with label $+2^{j'}$, we have $P(x+2^{i'}, w_P)_{j'} = 0$ and $P(w'_P, 2^i)_{j'} = 1$. Moreover, since $P(x+2^{i'}, w_P)$ never changes a bit from 1 to 0, we have $P(x+2^{i'}, w_P)_\ell = 1$ and $P(w'_P, 2^i)_\ell = 0$. On the other hand, since $x_{j'}$ has been changed to 1 when Q passes through the first edge and then keeps unchanged until Q passes through the last edge, we have $Q(x+2^{j'}, 2^{n-k}-1)_{j'} = 1$. Moreover, since $Q(x+2^{j'}, 2^{n-k}-1)$ never changes a bit from 1 to 0, we have $Q(x+2^{j'}, 2^{n-k}-1)_\ell = 1$. Thus, there is a different bit between nodes of $P(x+2^{i'}, w_P) \cup P(w'_P, 2^i)$ and $Q(x+2^{j'}, 2^{n-k}-1)$.

Case 3: $H^1_{n,n-k}(x) \neq \emptyset$ and $0 < |H^0_{n-k}(x)| < (n-k)/2-\lambda$. In this case, P is the same as that described in Case 2. From the paths constructed in Claim 1.3 (cf. Eq. (15)), Q starts with an edge labeled by $-2^{j'}$, where $j' = \max H^1_n(x)$, and ends with an edge labeled by $*$. Since $H^1_{n,n-k}(x) \neq \emptyset$, we have $j' \in H^1_{n,n-k}(x)$. Moreover, since $|H^1_{n-k}(x)| > (n-k)/2+\lambda$, there is a position $\ell \in H^1_{n-k}(x)$ such that $x_\ell = 1$. Since $P(x+2^{i'}, w_P)_\ell = 1$, it implies $P(w'_P, 2^i)_\ell = 0$. Also, since $P(x+2^{i'}, w_P)$ never changes a bit from 1 to 0, we have $P(x+2^{i'}, w_P)_{j'} = 1$. On the other hand, since $x_{j'}$ has been changed to 0 when Q passes through the first edge and then keeps unchanged until Q passes through the last edge, we have $Q(x-2^{j'}, 2^{n-k}-1)_{j'} = 0$. Moreover, since $Q(x-2^{j'}, 2^{n-k}-1)$ does not contain an edge with label -2^ℓ, we have $Q(x-2^{j'}, 2^{n-k}-1)_\ell = 1$. Thus, there is a different bit between nodes of $P(x+2^{i'}, w_P) \cup P(w'_P, 2^i)$ and $Q(x-2^{j'}, 2^{n-k}-1)$.

Case 4: $H^0_{n-k}(x) = \emptyset$. This case is impossible because $i \in H^0_{n-k}(x)$. □

Claim 2.13. *If $i \in H^1_{n,n-k}$ and $j = *$, then $P||Q$.*

Proof. Since $i \in H^1_{n,n-k}(x)$, it implies $H^1_{n,n-k}(x) \neq \emptyset$. There are six cases as follows.

Case 1: $H^1_{i,n-k}(x) \neq \emptyset$ and $|H^1_{n-k}(x)| < (n-k)/2$. This implies $|H^0_{n-k}(x)| > (n-k)/2$, and thus there is a position $\ell \in H^0_{n-k}(x)$ such that $x_\ell = 0$. From the paths constructed in Claim 1.2 (cf. Eq (9)), P starts with an edge labeled by $-2^{i'}$, where $i' = \max H^1_i(x)$, and ends with an edge labeled by -2^i. Also, from the paths constructed in Claim 1.3 (cf. Eq. (13)), Q has the label $*$ in its first edge and last edge. Since $P(x-2^{i'}, 2^i)$ never changes a bit from 0 to 1, we have $P(x-2^{i'}, 2^i)_\ell = 0$. On the other hand, since x_ℓ has been changed to 1 when Q passes through the first edge and then keeps unchanged until Q passes through the last edge, we have $Q(x \oplus (2^{n-k}-1), 2^{n-k}-1)_\ell = 1$. Thus, $P(x-2^{i'}, 2^i) \cap Q(x \oplus (2^{n-k}-1), 2^{n-k}-1) = \emptyset$.

Case 2: $H^1_{i,n-k}(x) = \emptyset$ and $|H^1_{n-k}(x)| < (n-k)/2$. This implies $|H^0_{n-k}(x)| > (n-k)/2$. In this case, P is a path constructed in Claim 1.2(cf. Eqs. (8) and (10)) and Q is a path constructed in Claim 1.3 (cf. Eq. (13)). The same argument as that in Case 1 shows that $P||Q$.

Case 3: $H^1_{i,n-k}(x) \neq \emptyset$ and $(n-k)/2 \leqslant |H^1_{n-k}(x)| \leqslant (n-k)/2 + \lambda$. It implies $(n-k)/2 - \lambda \leqslant |H^0_{n-k}(x)| \leqslant (n-k)/2$. In this case, P is the same as that described in Case 1. From the paths constructed in Claim 1.3 (cf. Eq. (14)), Q starts with an edge labeled by $+2^{j'}$, where $j' = \max H^0_{n-k}(x)$, and ends with an edge labeled by $*$. Since $P(x-2^{i'}, 2^i)$ never changes a bit from 0 to 1, we have $P(x-2^{i'}, 2^i)_{j'} = 0$. On the other hand, since $x_{j'}$ has been changed to 1 when Q passes through the first edge and then keeps unchanged until Q passes through the last edge, we have $Q(x+2^{j'}, 2^{n-k}-1)_{j'} = 1$. Thus, $P(x-2^{i'}, 2^i) \cap Q(x+2^{j'}, 2^{n-k}-1) = \emptyset$.

Case 4: $H^1_{i,n-k}(x) = \emptyset$ and $(n-k)/2 \leqslant |H^1_{n-k}(x)| \leqslant (n-k)/2+\lambda$. This implies $(n-k)/2-\lambda \leqslant |H^0_{n-k}(x)| \leqslant (n-k)/2$. In this case, P is a path constructed in Claim 1.2 (cf. Eq. (10)) and Q is a path constructed in Claim 1.3 (cf. Eq. (14)). The same argument as that in Case 3 shows that $P||Q$.

Case 5: $H^1_{i,n-k}(x) \neq \emptyset$ and $|H^1_{n-k}(x)| > (n-k)/2+\lambda$. This implies $|H^0_{n-k}(x)| < (n-k)/2 - \lambda$. From the paths constructed in Claim 1.2 (cf. Eq (9)), P starts with an edge labeled by $-2^{i'}$, where $i' = \max H^1_i(x)$, and ends with an edge labeled by -2^i. Since $x_{i'}$ has been changed to 0 when P passes through the first edge, we have $P(x-2^{i'}, 2^i)_{i'} = 0$. Also, since x_i remains unchanged until P passes through the last edge, we have $P(x-2^{i'}, 2^i)_i = 1$. On the other hand, from the paths constructed in Claim 1.3 (cf. Eq. (15)), Q starts with an edge labeled by $-2^{j'}$, where $j' = \max^1_n(x)$, and ends with an edge labeled by $*$. Note that it is possible $j' = i$. If $j' \neq i$, it is clear that $Q(x-2^{j'}, 2^{n-k}-1)$ contains an edge with label -2^i, denoted by $w \xrightarrow{-2^i} w'$. Since $x_{i'}$ alters after the change of x_i in Q, we have $Q(x-2^{j'}, w)_{i'} = 1$ and $Q(w', 2^{n-k}-1)_i = 0$. This shows that every node of $P(x-2^{i'}, 2^i)$ has a bit different from nodes of $Q(x-2^{j'}, w) \cup Q(w', 2^{n-k}-1)$.

Case 6: $H^1_{i,n-k}(x) = \emptyset$ and $|H^1_{n-k}(x)| > (n-k)/2 + \lambda$. This implies $|H^0_{n-k}(x)| < (n-k)/2 - \lambda$. Since $|H^1_{n-k}(x)| > 1$, there is a position $\ell \in H^1_{n-k}(x)$ such that $x_\ell = 1$. In this case, Q is the same as that described in Case 5. Since $x_{j'}$ has been changed to 0 when Q passes through the first edge and then keeps unchanged until Q passes through the last edge, we have $Q(x-2^{j'}, 2^{n-k}-1)_{j'} = 0$. Also, since $Q(x-2^{j'}, 2^{n-k}-1)$ never changes x_ℓ to 0, we have $Q(x-2^{j'}, 2^{n-k}-1)_\ell = 1$. Let $i' = \textsc{next}_x(i)$. From the paths constructed in Claim 1.2 (cf. Eqs. (11) and (12)), P starts with an edge labeled by $+2^{i'}$ or $*$, and ends with an edge labeled by -2^i. If $i' \neq *$, then $i' = \max H^0_{n-k}(x)$ and P must have passed through an edge with label $*$, denoted by $w \xrightarrow{*} w'$. Since $P(x+2^{i'}, w)$ never changes a bit from 1 to 0, we have $P(x+2^{i'}, w)_\ell = 1$ and $P(w', 2^i)$ contains an edge with label $-2^{j'}$. This further implies that $P(x+2^{i'}, w)_{j'} = 1$ and $P(w', 2^i)_\ell = 0$. This shows that every node of $P(x+2^{i'}, w) \cup P(w', 2^i)$ has a bit different from nodes of $Q(x-2^{j'}, 2^{n-k}-1)$. □

Claim 2.14. *If $i \in H^0_{n,n-k}$ and $j = *$, then $P||Q$.*

Proof. Note that $x_i = 0$. From the paths constructed in Claim 1.2 (cf. Eq. (7)), we have $\textsc{next}_x(i) = i$. Thus, P has the label $+2^i$ in its first edge and the label -2^i in its last edge. Since x_i has been changed to 1 when P passes through the first edge and then keeps unchanged until P passes through the last edge, we have $P(x+2^i, 2^i)_i = 1$. On the other hand, let w be the node adjacent to x in Q. From the paths constructed in Claim 1.3 (cf. Eqs. from (13) to (18)), Q never changes a bit in $H^0_{n,n-k}(x)$ from 0 to 1. Since $i \in H^0_{n,n-k}(x)$, we have $Q(w, 2^{n-k}-1)_i = 0$. Thus, $P(x+2^i, 2^i) \cap Q(w, 2^{n-k}-1) = \emptyset$. □