DM-05-01-b ► 解遞迴關係式

$$\begin{cases} a_n - 2a_{n-1} + 2a_{n-2} - a_{n-3} = 0, & n \geq 3 \\ a_0 = 3, \quad a_1 = 2, \quad a_2 = 2 \end{cases}$$

【解】 特徵方程式為 $\alpha^3 - 2\alpha^2 + 2\alpha - 1 = 0$，即 $(\alpha-1)(\alpha^2-\alpha+1) = (\alpha-1)(\alpha-\frac{1+\sqrt{3}i}{2})(\alpha-\frac{1-\sqrt{3}i}{2}) = 0$。因此 $\alpha_1 = 1$，$\alpha_2 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$， $\alpha_3 = \frac{1}{2} - \frac{\sqrt{3}}{2}i$ 為特徵根。
故遞迴式之一般解為

$$a_n = c_1 \cdot (1)^n + c_1 \cdot (\alpha_1)^n + c_2 \cdot (\alpha_2)^n = k_0 + c_1 \cdot \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^n + c_2 \cdot \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^n, \quad n \geq 0$$

其中 k_0, c_1 與 c_2 為複數常數。 因 $\cos\frac{\pi}{3} = \cos(-\frac{\pi}{3}) = \frac{1}{2}$，$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$， $\sin(-\frac{\pi}{3}) = -\frac{\sqrt{3}}{2}$，前述遞迴式之一般解亦可表示如下：

$$a_n = k_0 + c_1 \cdot \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^n + c_2 \cdot \left(\cos(-\frac{\pi}{3}) + i\sin(-\frac{\pi}{3})\right)^n, \quad n \geq 0$$

由隸美佛公式 (DeMoivre's Theorem) $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$ 得

$$\begin{aligned} a_n &= k_0 + c_1 \cdot \left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right) + c_2 \cdot \left(\cos\frac{-n\pi}{3} + i\sin\frac{-n\pi}{3}\right) \\ &= k_0 + c_1 \cdot \left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right) + c_2 \cdot \left(\cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right) \\ &= k_0 + (c_1 + c_2) \cdot \cos\frac{n\pi}{3} + (c_1 - c_2)i \cdot \sin\frac{n\pi}{3} \\ &= k_0 + k_1 \cdot \cos\frac{n\pi}{3} + k_2 \cdot \sin\frac{n\pi}{3} \end{aligned}$$

其中 $k_1 = c_1 + c_2$ 且 $k_2 = (c_1 - c_2)i$。代入起始條件 $a_0 = 3$，$a_1 = 2$ 與 $a_2 = 2$ 得

$$\begin{cases} k_0 + k_1 \cdot \cos 0 + k_2 \cdot \sin 0 = k_0 + k_1 = 3 \\ k_0 + k_1 \cdot \cos\frac{\pi}{3} + k_2 \cdot \sin\frac{\pi}{3} = k_0 + \frac{1}{2}k_1 + \frac{\sqrt{3}}{2}k_2 = 2 \\ k_0 + k_1 \cdot \cos\frac{2\pi}{3} + k_2 \cdot \sin\frac{2\pi}{3} = k_0 - \frac{1}{2}k_1 + \frac{\sqrt{3}}{2}k_2 = 2 \end{cases}$$

解聯立方程式得 $k_0 = 3$，$k_1 = 0$ 與 $k_2 = -\frac{2}{\sqrt{3}}$。即

$$\begin{cases} c_1 + c_2 = 0 \\ (c_1 - c_2)i = -\frac{2}{\sqrt{3}} \end{cases}$$

再次解聯立方程式得 $c_1 = \frac{1}{\sqrt{3}}i$ 且 $c_2 = -\frac{1}{\sqrt{3}}i$。
故

$$\{a_n\}_{n=0}^{\infty} = \left\{3 + (\frac{1}{\sqrt{3}}i) \cdot (\frac{1}{2} + \frac{\sqrt{3}}{2}i)^n + (-\frac{1}{\sqrt{3}}i) \cdot (\frac{1}{2} - \frac{\sqrt{3}}{2}i)^n\right\}_{n=0}^{\infty}$$

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