▶ **Problem DM-1.11-12** The terms of a sequence are given recursively as $p_0 = 3$, $p_1 = 7$, and $p_n = 3p_{n-1} - 2p_{n-2}$ for $n \geq 2$. Prove by induction that $b_n = 2^{n+2} - 1$ is a closed form for the sequence.

Proof. Let $n_0 = 0$ and $\mathcal{T} = \{n \in \mathbb{N} : b_n = p_n\}$.

(Base step) Identify $n = 0, 1$ as the base cases. Evaluate b_0 and b_1 directly:

$$b_0 = 2^{0+2} - 1 = 2^2 - 1 = 3 = p_0$$

$$b_1 = 2^{1+2} - 1 = 2^3 - 1 = 7 = p_1$$

So, $0, 1 \in \mathcal{T}$.

(Inductive step) Now, let $n \geq 2$ and assume for $k = 0, 1, \ldots, n-1$ that $k \in \mathcal{T}$. Prove that $n \in \mathcal{T}$ by showing that $p_n = b_n$ as follows.

$$\begin{aligned} p_n &= 3p_{n-1} - 2p_{n-2} \\ &= 3 \cdot (2^{(n-1)+2} - 1) - 2 \cdot (2^{(n-2)+2} - 1) \\ &= 3 \cdot 2^{n+1} - 3 - 2 \cdot 2^n + 2 \\ &= 3 \cdot 2^{n+1} - 2^{n+1} - 1 \\ &= 2 \cdot 2^{n+1} - 1 \\ &= 2^{n+2} - 1 \\ &= b_n \end{aligned}$$

Therefore, $n \in \mathcal{T}$.

By the Strong form of Mathematical Induction, $\mathcal{T} = \mathbb{N}$. That is, $b_n = 2^{n+2} - 1$ is a closed form for the terms of the recursively defined sequence.

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