▶ **Problem DM-1.11-16** Prove by induction that

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

is a closed form for the Fibonacci sequence.

Proof. Let $n_0 = 0$ and

$$\mathcal{T} = \{n \in \mathbb{N} : F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\}.$$

(Base step) Identify $n = 0, 1$ as the base cases. Evaluate F_0 and F_1 directly:

$$F_0 = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right) - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right) = 1$$

$$F_1 = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^2 - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{1}{\sqrt{5}}\left(\frac{6+2\sqrt{5}}{4} - \frac{6-2\sqrt{5}}{4}\right) = 1$$

So, $0, 1 \in \mathcal{T}$.

(Inductive step) Now, let $n \geq 2$ and assume for $k = 0, 1, \ldots, n-1$ that $k \in \mathcal{T}$. Prove that $n \in \mathcal{T}$ by showing that F_n has the desired closed form. Note that $1 + \frac{2}{1+\sqrt{5}} = \frac{1+\sqrt{5}}{2}$ and $1 + \frac{2}{1-\sqrt{5}} = \frac{1-\sqrt{5}}{2}$. Then

$$\begin{aligned}
F_n &= F_{n-1} + F_{n-2} \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] + \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right] \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\right] - \frac{1}{\sqrt{5}}\left[\left(\frac{1-\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right] \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n\left(1 + \frac{2}{1+\sqrt{5}}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^n\left(1 + \frac{2}{1-\sqrt{5}}\right)\right] \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]
\end{aligned}$$

Therefore, $n \in \mathcal{T}$.

By the Strong form of Mathematical Induction, $\mathcal{T} = \mathbb{N}$. That is,

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

is a closed form for the terms of the recursively defined sequence. □