▶ **Problem DM-1.4-16** For (a) and (b), prove the stated result. For (c) and (d), find a counterexample to show that these conjectures are false.

(a) $A \oplus B = (A \cup B) - (A \cap B)$

(b) $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$

(c) $(A \cap B) \oplus (C \cap D) \subseteq (A \oplus C) \cap (B \oplus D)$

(d) $(A \cup B) \oplus (C \cup D) \subseteq (A \cup C) \oplus (B \cup D)$

Proof. (a)

$$
\begin{aligned}
& A \oplus B \\
= \;& (A - B) \cup (B - A) \qquad \text{// Definition 7} \\
= \;& \{x : x \in A \text{ and } x \notin B\} \cup \{x : x \in B \text{ and } x \notin A\} \qquad \text{// Definition 5} \\
= \;& (A \cap \overline{B}) \cup (B \cap \overline{A}) \\
= \;& ((A \cap \overline{B}) \cup B) \cap ((A \cap \overline{B}) \cup \overline{A}) \qquad \text{// Distributive Law for Union} \\
= \;& ((A \cup B) \cap (\overline{B} \cup B)) \cap (\overline{A} \cup A) \cap (\overline{B} \cup \overline{A})) \\
= \;& ((A \cup B) \cap U) \cap (U \cap (\overline{A} \cup \overline{B})) \\
= \;& (A \cup B) \cap (\overline{A} \cup \overline{B}) \\
= \;& (A \cup B) \cap \overline{(A \cap B)} \qquad \text{// DeMorgan's Law for Intersection} \\
= \;& \{x : x \in A \cup B \text{ and } x \notin A \cap B\} \\
= \;& (A \cup B) - (A \cap B) \qquad \text{// Definition 5}
\end{aligned}
$$

(b)

$$
\begin{aligned}
& A \cap (B \oplus C) \\
= \;& A \cap ((B - C) \cup (C - B)) \qquad \text{// Definition 7} \\
= \;& A \cap ((B \cap \overline{C}) \cup (C \cap \overline{B})) \qquad \text{// Definition 5} \\
= \;& (A \cap (B \cap \overline{C})) \cup (A \cap (C \cap \overline{B})) \qquad \text{// Distributive Law for Intersection} \\
= \;& (A \cap B \cap \overline{C}) \cup (A \cap C \cap \overline{B}) \\
= \;& (A \cap B \cap \overline{A}) \cup (A \cap B \cap \overline{C}) \cup (A \cap C \cap \overline{A}) \cup (A \cap C \cap \overline{B}) \;\; \text{// Since } A \cap \overline{A} = \emptyset \\
= \;& ((A \cap B \cap \overline{A}) \cup (A \cap B \cap \overline{C})) \cup ((A \cap C \cap \overline{A}) \cup (A \cap C \cap \overline{B})) \\
= \;& ((A \cap B) \cap (\overline{A} \cup \overline{C})) \cup ((A \cap C) \cap (\overline{A} \cup \overline{B})) \;\; \text{// Distributive Law for Intersection} \\
= \;& ((A \cap B) \cap \overline{(A \cap C)}) \cup ((A \cap C) \cap \overline{(A \cap B)}) \;\; \text{// DeMorgan's Law for Intersection} \\
= \;& ((A \cap B) - (A \cap C)) \cup ((A \cap C) - (A \cap B)) \\
= \;& (A \cap B) \oplus (A \cap C) \qquad \text{// Definition 7}
\end{aligned}
$$

(c) Let $A = \{2, 3\}$, $B = \{1, 3\}$, $C = \{1, 4\}$, and $D = \{3, 4\}$. Then, $(A \cap B) \oplus (C \cap D) = \{3\} \oplus \{4\} = \{3, 4\}$. However, $(A \oplus C) \cap (B \oplus D) = \{1, 2, 3, 4\} \cap \{1, 4\} = \{1, 4\}$. This shows that $(A \cap B) \oplus (C \cap D) \not\subseteq (A \oplus C) \cap (B \oplus D)$.

(d) Let $A = \{1\}$, $B = \{1, 2\}$, $C = \{1, 2, 3\}$, and $D = \{1, 2, 3, 4\}$. Then, $(A \cup B) \oplus (C \cup D) = \{1, 2\} \oplus \{1, 2, 3, 4\} = \{3, 4\}$. However, $(A \cup C) \oplus (B \cup D) = \{1, 2, 3\} \oplus \{1, 2, 3, 4\} = \{4\}$. This shows that $(A \cup B) \oplus (C \cup D) \not\subseteq (A \cup C) \oplus (B \cup D)$. □