▶ **Problem DM-1.4-24** Recall that in the definition of a boolean algebra, we did not require that $\top$, $\perp$, and each $\neg x$ *be specified*; we merely said they must exist. So, it is natural to ask whether there might be several elements that could equally well be chosen as $\top$ or $\perp$ or, for some element x of the boolean algebra, several different possible choices for $\neg x$. Show that in a complemented lattice:

(a) There is only one possible choice of elements $\top$ and $\perp$ satisfying the definition of a complemented lattice.

(b) For each element x of a complemented, distributive lattice, there is only one possible choice for $\neg x$ that satisfies the definition of $\neg x$.

Proof. (a) Consider a complemented lattice X together with the operations $\wedge$ and $\vee$. From the definition, we have $x \wedge \top = x$ for every $x \in X$. Suppose that there exist two possible choices for $\top$, say $\top_1$ and $\top_2$. We now evaluate $\top_1 \wedge \top_2$ in two different ways: (i) Since $\top_2$ is maximum element, $\top_1 \wedge \top_2 = \top_1$, and (ii) By the commutative law for meet and since $\top_1$ is maximum element, $\top_1 \wedge \top_2 = \top_2 \wedge \top_1 = \top_2$. Thus, if $\top_1 \neq \top_2$, then it leads to a contradiction. Therefore, the maximum element $\top$ in X is unique.

The fact that the minimum element $\perp$ in X is unique can be proved by a similar way.

(b) Suppose that X together with $\wedge$ and $\vee$ form a boolean algebra (i.e., a complemented, distributive lattice) and $x \in X$. To show that $\neg x$ is unique in X, we suppose to the contrary that there exist two choices, say $\neg x_1$ and $\neg x_2$, for $\neg x$. We now evaluate $\neg x_1 \wedge x \vee \neg x_2$ in two different ways:

$$\begin{aligned}
\neg x_1 \wedge x \vee \neg x_2 &= (\neg x_1 \wedge x) \vee \neg x_2 \\
&= \perp \vee \neg x_2 \\
&= \neg x_2
\end{aligned}$$

and

$$\begin{aligned}
\neg x_1 \wedge x \vee \neg x_2 &= (\neg x_1 \wedge x) \vee \neg x_2 \\
&= (\neg x_1 \vee \neg x_2) \wedge (x \vee \neg x_2) \qquad \text{// Distributive Law for Join} \\
&= (\neg x_1 \vee \neg x_2) \wedge \top \\
&= \neg x_1 \vee \neg x_2
\end{aligned}$$

This shows that $\neg x_2 = \neg x_1 \vee \neg x_2$. On the other hand, if we evaluate $\neg x_2 \wedge x \vee \neg x_1$ in two different ways similarly, we can obtain $\neg x_1 = \neg x_2 \vee \neg x_1$. Therefore, the element $\neg x_1 = \neg x_2 \vee \neg x_1 = \neg x_1 \vee \neg x_2 = \neg x_2$ is unique. $\square$