▶ **Problem DM-1.6-16** Determine how many numbers between 1 and 21,000,000,000, including 1 and 21,000,000,000, are divisible by 2, 3, 5, or 7.

**Solution.** Let $D_i = \{n \in \mathbb{N} : 1 \leq n \leq 21,000,000,000 \text{ and } n \text{ is divisible by } i\}$. According to the problem, we need to compute $|D_2 \cup D_3 \cup D_5 \cup D_7|$. By the Principle of Inclusion-Exclusion, we have

$$\begin{aligned}|D_2 \cup D_3 \cup D_5 \cup D_7| &= |D_2| + |D_3| + |D_5| + |D_7| - |D_2 \cap D_3| - |D_2 \cap D_5| - \\ &\quad |D_2 \cap D_7| - |D_3 \cap D_5| - |D_3 \cap D_7| - |D_5 \cap D_7| + \\ &\quad |D_2 \cap D_3 \cap D_5| + |D_2 \cap D_3 \cap D_7| + |D_2 \cap D_5 \cap D_7| + \\ &\quad |D_3 \cap D_5 \cap D_7| - |D_2 \cap D_3 \cap D_5 \cap D_7|\end{aligned}$$

Obviously, $|D_2| = 10,500,000,000$, $|D_3| = 7,000,000,000$, $|D_5| = 4,200,000,000$, and $|D_7| = 3,000,000,000$. Since 2 and 3 are both prime, an integer $n$ is divisible by both 2 and 3 if and only if $n$ is divisible by $2 \cdot 3 = 6$. So $D_2 \cap D_3 = D_6$ and $|D_6| = 3,500,000,000$. Similarly,

$$|D_2 \cap D_5| = |D_{10}| = 2,100,000,000$$

$$|D_2 \cap D_7| = |D_{14}| = 1,500,000,000$$

$$|D_3 \cap D_5| = |D_{15}| = 1,400,000,000$$

$$|D_3 \cap D_7| = |D_{21}| = 1,000,000,000$$

$$|D_5 \cap D_7| = |D_{35}| = 600,000,000$$

$$|D_2 \cap D_3 \cap D_5| = |D_{30}| = 700,000,000$$

$$|D_2 \cap D_3 \cap D_7| = |D_{42}| = 500,000,000$$

$$|D_2 \cap D_5 \cap D_7| = |D_{70}| = 300,000,000$$

$$|D_3 \cap D_5 \cap D_7| = |D_{105}| = 200,000,000$$

and

$$|D_2 \cap D_3 \cap D_5 \cap D_7| = |D_{210}| = 100,000,000$$

Now, by the Principle of Inclusion-Exclusion,

$$\begin{aligned}|D_2 \cup D_3 \cup D_5 \cup D_7| &= 10,500,000,000 + 7,000,000,000 + 4,200,000,000 + \\ &\quad 3,000,000,000 - 3,500,000,000 - 2,100,000,000 - \\ &\quad 1,500,000,000 - 1,400,000,000 - 1,000,000,000 - \\ &\quad 600,000,000 + 700,000,000 + 500,000,000 + \\ &\quad 300,000,000 + 200,000,000 - 100,000,000 \\ &= 16,200,000,000\end{aligned}$$

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