▶ **Problem DM-1.6-20** Find the number of integers between 1 and 1000, including 1 and 1000, that are not divisible by any of 4, 6, 7, or 10.

Solution. Let $U = \{n \in \mathbb{N} : 1 \leq n \leq 1000\}$ and $D_i = \{n \in \mathbb{N} : 1 \leq n \leq 1000$ and n is divisible by $i\}$. According to the problem, we need to compute $|\overline{D_4 \cup D_6 \cup D_7 \cup D_{10}}|$. By the Principle of Inclusion-Exclusion, we have

$$\begin{aligned} |D_4 \cup D_6 \cup D_7 \cup D_{10}| &= |D_4| + |D_6| + |D_7| + |D_{10}| - |D_4 \cap D_6| - |D_4 \cap D_7| - \\ &\quad |D_4 \cap D_{10}| - |D_6 \cap D_7| - |D_6 \cap D_{10}| - |D_7 \cap D_{10}| + \\ &\quad |D_4 \cap D_6 \cap D_7| + |D_4 \cap D_6 \cap D_{10}| + |D_4 \cap D_7 \cap D_{10}| + \\ &\quad |D_6 \cap D_7 \cap D_{10}| - |D_4 \cap D_6 \cap D_7 \cap D_{10}| \end{aligned}$$

and $|\overline{D_4 \cup D_6 \cup D_7 \cup D_{10}}| = |U| - |D_4 \cup D_6 \cup D_7 \cup D_{10}|$.

Obviously, $|U| = 1000$, $|D_4| = 250$, $|D_6| = 166$, $|D_7| = 142$, and $|D_{10}| = 100$. Also, for any two integers a and b, an integer n is divisible by both a and b if and only if n is divisible by $LCM(a, b)$, the least common multiple of a and b. So $D_4 \cap D_6 = D_{12}$ and $|D_{12}| = 83$. Similarly,

$$|D_4 \cap D_7| = |D_{28}| = 35$$

$$|D_4 \cap D_{10}| = |D_{20}| = 50$$

$$|D_6 \cap D_7| = |D_{42}| = 23$$

$$|D_6 \cap D_{10}| = |D_{30}| = 33$$

$$|D_7 \cap D_{10}| = |D_{70}| = 14$$

$$|D_4 \cap D_6 \cap D_7| = |D_{84}| = 11$$

$$|D_4 \cap D_6 \cap D_{10}| = |D_{60}| = 16$$

$$|D_4 \cap D_7 \cap D_{10}| = |D_{140}| = 7$$

$$|D_6 \cap D_7 \cap D_{10}| = |D_{210}| = 4$$

and

$$|D_4 \cap D_6 \cap D_7 \cap D_{10}| = |D_{420}| = 2$$

Now, by the Principle of Inclusion-Exclusion,

$$\begin{aligned} &|\overline{D_4 \cup D_6 \cup D_7 \cup D_{10}}| \\ &= 1000 - (250 + 166 + 142 + 100 - 83 - 35 - 50 - 23 - 33 - 14 + 11 + 16 + 7 + 4 - 2) \\ &= 544 \end{aligned}$$

□