▶ **Problem DM-1.9-22** Show that any integer consisting of 3^n identical digits is divisible by 3^n. Verify this for 222; 777; 222, 222, 222; and 555, 555, 555. Prove the general statement for all $n \in \mathbb{N}$ by induction.

Verification: For integers a and b, we say a is a divisor of b, denoted as $a|b$, if there is a nutural number k such that $b = a \cdot k$. For $n = 1$, it is easy to check that $222/3^1 = 74$ and $777/3^1 = 259$. Thus, we have $3^1|222$ and $3^1|777$. For $n = 2$, since $222,222,222/3^2 = 24,691,358$ and $555,555,555/3^2 = 61,728,395$, we have $3^2|222,222,222$ and $3^2|555,555,555$.

We now show that $3^n| \underbrace{xx\cdots x}_{\text{the length is } 3^n}$ for any digit $x \in \{0,1,2,3,4,5,6,7,8,9\}$ by induction.

Proof. Let $n_0 = 0$ and

$$\mathcal{T} = \{n \in \mathbb{N} : 3^n|X \text{ where } X = xx\cdots x \text{ is an integer consisting of } 3^n \text{ identical digits}\}.$$

(Base step) For $n = 0$, it is obvious that $3^0 = 1$ is a divisor of any integer $X \in \{0,1,2,3,4,5,6,7,8,9\}$. Thus, $0 \in \mathcal{T}$.

(Inductive step) Let $n \geq 0$. Show that if $n \in \mathcal{T}$, then $n+1 \in \mathcal{T}$. Since $n \in \mathcal{T}$, it is assumed that $3^n|X$ where $X = xx\cdots x$ is an integer consisting of 3^n identical digits x. We will prove that $3^{n+1}|Y$ where $Y = xx\cdots x$ is an integer consisting of 3^{n+1} identical digits x.

Let $x \in \{0,1,2,3,4,5,6,7,8,9\}$ be any digit, and let X and Y be two integers consisting of 3^n and 3^{n+1} identical digits x, respectively. Then

$$Y = X \cdot \left[(10^{3^n})^2 + (10^{3^n})^1 + (10^{3^n})^0\right] \tag{1}$$

For instance, if $n = 0$ and $X = 2$, then $Y = 2 \cdot (10^2 + 10^1 + 10^0) = 222$. Also, if $n = 1$ and $X = 222$, then $Y = 222 \cdot (10^6 + 10^3 + 10^0) = 222,222,222$.

By induction hypothesis, we have $3^n|X$. Also, it is clear that $3|(10^{3^n})^2 + (10^{3^n})^1 + (10^{3^n})^0$. Therefore, by (1) we conclude that $3^{n+1}|Y$. □