▶ **Problem DM-1.9-24(a)-(b)** Prove the following results for the Fibonacci numbers:

(a) F_{3n} and F_{3n+1} are odd, and F_{3n+2} is even for $n \geq 0$.

(b) $F_0 + F_2 + \cdots + F_{2n} = F_{2n+1}$ for $n \geq 0$.

Proof. (a) Let $n_0 = 0$ and $\mathcal{T} = \{n \in \mathbb{N} : F_{3n}$ and F_{3n+1} are odd and F_{3n+2} is even$\}$. We will prove by induction that $\mathcal{T} = \mathbb{N}$.

(Base step) By the definition of Fibonacci numbers, $F_0 = F_1 = 1$ is odd and $F_2 = F_0 + F_1 = 2$ is even. So, $0 \in \mathcal{T}$.

(Inductive step) Let $n \geq 0$. Show that if $n \in \mathcal{T}$, then $n+1 \in \mathcal{T}$. Since $n \in \mathcal{T}$, it is assumed that F_{3n} and F_{3n+1} are odd, and F_{3n+2} is even. We must prove that $F_{3(n+1)}$ and $F_{3(n+1)+1}$ are odd, and $F_{3(n+1)+2}$ is even. We are now easy to verify as follows:

$F_{3(n+1)} = F_{3n+3} = F_{3n+1} + F_{3n+2}$ is odd (since odd + even = odd)

$F_{3(n+1)+1} = F_{3n+4} = F_{3n+2} + F_{3n+3}$ is odd (since even + odd = odd)

and

$F_{3(n+1)+2} = F_{3n+5} = F_{3n+3} + F_{3n+4}$ is even (since odd + odd = even)

Therefore, $n+1 \in \mathcal{T}$.

By the Principle of Mathematical Induction, $\mathcal{T} = \mathbb{N}$. □

(b) Let $n_0 = 0$ and $\mathcal{T} = \{n \in \mathbb{N} : F_0 + F_2 + \cdots + F_{2n} = F_{2n+1}\}$. We will prove by induction that $\mathcal{T} = \mathbb{N}$.

(Base step) For $n = 0$, the left-hand side is $F_0 = 1$ and the right-hand side is $F_{2\cdot 0+1} = F_1 = 1$. So, the two sides are equal, and $0 \in \mathcal{T}$.

(Inductive step) Let $n \geq 0$. Show that if $n \in \mathcal{T}$, then $n+1 \in \mathcal{T}$. Since $n \in \mathcal{T}$, it is assumed that $F_0+F_2+\cdots+F_{2n} = F_{2n+1}$. We must prove that $F_0+F_2+\cdots+F_{2n}+F_{2(n+1)} = F_{2(n+1)+1}$. The required computation is

$$
\begin{aligned}
& F_0 + F_2 + \cdots + F_{2n} + F_{2(n+1)} \\
= \; & (F_0 + F_2 + \cdots + F_{2n}) + F_{2n+2} \\
= \; & F_{2n+1} + F_{2n+2} \\
= \; & F_{2n+3} \\
= \; & F_{2(n+1)+1}
\end{aligned}
$$

Therefore, $n+1 \in \mathcal{T}$.

By the Principle of Mathematical Induction, $\mathcal{T} = \mathbb{N}$. □