▶ **Problem DM-1.9-24(c)-(d)** Prove the following results for the Fibonacci numbers:.

(c) $F_0 + F_3 + \cdots + F_{3n} = F_{3n+2}/2$ for $n \geq 0$.

(d) $F_{n+1}^2 = F_n \cdot F_{n+2} - (-1)^n$ for $n \geq 0$.

Proof. (c) Let $n_0 = 0$ and $\mathcal{T} = \{n \in \mathbb{N} : F_0 + F_3 + \cdots + F_{3n} = \frac{1}{2}F_{3n+2}\}$. We will prove by induction that $\mathcal{T} = \mathbb{N}$.

(Base step) For $n = 0$, the left-hand side is $F_0 = 1$ and the right-hand side is $\frac{1}{2}F_{3\cdot 0+2} = \frac{1}{2}F_2 = 2/2 = 1$. So, the two sides are equal, and $0 \in \mathcal{T}$.

(Inductive step) Let $n \geq 0$. Show that if $n \in \mathcal{T}$, then $n + 1 \in \mathcal{T}$. Since $n \in \mathcal{T}$, it is assumed that $F_0 + F_3 + \cdots + F_{3n} = \frac{1}{2}F_{3n+2}$. We must prove that $F_0 + F_3 + \cdots + F_{3n} + F_{3(n+1)} = \frac{1}{2}F_{3(n+1)+2}$. The required computation is

$$\begin{aligned}
& F_0 + F_3 + \cdots + F_{3n} + F_{3(n+1)} \\
= \; & (F_0 + F_3 + \cdots + F_{3n}) + F_{3n+3} \\
= \; & \frac{1}{2}F_{3n+2} + F_{3n+3} \\
= \; & \frac{1}{2}(F_{3n+2} + 2F_{3n+3}) \\
= \; & \frac{1}{2}\big((F_{3n+2} + F_{3n+3}) + F_{3n+3}\big) \\
= \; & \frac{1}{2}(F_{3n+4} + F_{3n+3}) \\
= \; & \frac{1}{2}F_{3n+5} \\
= \; & \frac{1}{2}F_{3(n+1)+2}
\end{aligned}$$

Therefore, $n + 1 \in \mathcal{T}$.

By the Principle of Mathematical Induction, $\mathcal{T} = \mathbb{N}$. □

(d) Let $n_0 = 0$ and $\mathcal{T} = \{n \in \mathbb{N} : F_{n+1}^2 = F_n \cdot F_{n+2} - (-1)^n\}$. We will prove by induction that $\mathcal{T} = \mathbb{N}$.

(Base step) For $n = 0$, the left-hand side is $F_1^2 = 1$ and the right-hand side is $F_0 \cdot F_2 - (-1)^0 = 1 \cdot 2 - 1 = 1$. So, the two sides are equal, and $0 \in \mathcal{T}$.

(Inductive step) Let $n \geq 0$. Show that if $n \in \mathcal{T}$, then $n+1 \in \mathcal{T}$. Since $n \in \mathcal{T}$, it is assumed that

$$F_{n+1}^2 = F_n \cdot F_{n+2} - (-1)^n.$$

We must prove that

$$F_{(n+1)+1}^2 = F_{n+1} \cdot F_{(n+1)+2} - (-1)^{(n+1)}.$$

The required computation is

$$\begin{aligned}
& F_{(n+1)+1}^2 \\
= \; & F_{n+2}^2 \\
= \; & F_{n+2} \cdot (F_n + F_{n+1}) \\
= \; & F_n \cdot F_{n+2} + F_{n+1} \cdot F_{n+2} \\
= \; & \big(F_n \cdot F_{n+2} - (-1)^n\big) + \big(F_{n+1} \cdot F_{n+2} + (-1)^n\big) \\
= \; & F_{n+1}^2 + \big(F_{n+1} \cdot F_{n+2} + (-1)^n\big) \\
= \; & F_{n+1} \cdot (F_{n+1} + F_{n+2}) - (-1)^{(n+1)} \\
= \; & F_{n+1} \cdot F_{n+3} - (-1)^{(n+1)} \\
= \; & F_{n+1} \cdot F_{(n+1)+2} - (-1)^{(n+1)}
\end{aligned}$$

Therefore, $n+1 \in \mathcal{T}$.

By the Principle of Mathematical Induction, $\mathcal{T} = \mathbb{N}$. □