▶ **Problem DM-3.3-10**

(a) Prove for any set X that $Id_X = Id_X^{-1}$.

(b) Find two binary relations R and S on $\mathbb{N}$ where $R \neq Id_{\mathbb{N}}$ and $S \neq Id_{\mathbb{N}}$ such that $R = R^{-1}$ and $S = S^{-1}$.

(c) Suppose that R is a binary relation on a set X and, for *every* binary relation S on X, $R \circ S = S$. Prove that $R = Id_X$.

Proof. (a) By definition,

$$\begin{aligned} Id_X &= \{(x, y) : x, y \in X \text{ and } x = y\} \\ &= \{(x, x) : x \in X\} \end{aligned}$$

Since $R^{-1} = \{(x, y) : (y, x) \in R\}$ for any binary relation R, we have

$$\begin{aligned} Id_X^{-1} &= \{(x, y) : (y, x) \in Id_X\} \\ &= \{(x, y) : y, x \in X \text{ and } y = x\} \\ &= \{(x, x) : x \in X\} \\ &= Id_X \end{aligned}$$

(b) Let $R = \{(x, x) : x \in \mathbb{N}$ is an odd integer$\}$ and $S = \{(y, y) : y \in \mathbb{N}$ is an even integer$\}$. Then, it is clear that $R = R^{-1} \neq \emptyset$ and $S = S^{-1} \neq \emptyset$. Since $Id_{\mathbb{N}} = R \cup S$, we have $R \neq Id_{\mathbb{N}}$ and $S \neq Id_{\mathbb{N}}$.

(c) Consider R to be a binary relation on a set X. To show that $R \circ S = S$ implies $R = Id_X$ for any relation S on X. We will prove that if $R \neq Id_X$ then there exists a relation S on X such that $R \circ S \neq S$. Since $R \neq Id_X$, there exist $x, y \in X$ with $x \neq y$ such that $(x, y) \in R$. We now let $S = Id_X$. Clearly, $(x, y) \notin S$ and $(x, x) \in S$. By the definition of composition $R \circ S$ and the the facts $(x, x) \in S$ and $(x, y) \in R$, we have $(x, y) \in R \circ S$. Therefore, $R \circ S \neq S$. □