▶ **Problem DM-3.5-12** Lex $X = \{1,2,3,4,5,6\}$, and define a relation R on X as

$$R = \{(1,2),(2,1),(2,3),(3,4),(4,5),(5,6)\}$$

(a) Find the reflexive closure of R.
(b) Find the symmetric closure of R.
(c) Find the transitive closure of R.
(d) Find the reflexive and transitive closure of R.

Solution.

$$Id_X = R^0 = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$$
$$R^{-1} = \{(1,2),(2,1),(3,2),(4,3),(5,4),(6,5)\}$$
$$R^2 = \{(1,1),(1,3),(2,2),(2,4),(3,5),(4,6)\}$$
$$R^3 = \{(1,2),(1,4),(2,1),(2,3),(2,5),(3,6)\}$$
$$R^4 = \{(1,1),(1,3),(1,5),(2,2),(2,4),(2,6)\}$$
$$R^5 = \{(1,2),(1,4),(1,6),(2,1),(2,3),(2,5)\}$$
$$R^6 = \{(1,1),(1,3),(1,5),(2,2),(2,4),(2,6)\} = R^4$$
$$R^7 = \{(1,2),(1,4),(1,6),(2,1),(2,3),(2,5)\} = R^5$$

In general, $R^{2n} = R^4$ and $R^{2n+1} = R^5$ for $n \geq 2$. Therefore,

(a) The reflexive closure of R is

$$\begin{aligned} & R \cup Id_X \\ = & \{(1,1),(1,2),(2,1),(2,2),(2,3),(3,3),(3,4),(4,4),(4,5),(5,5),(5,6),(6,6)\} \end{aligned}$$

(b) The symmetric closure of R is

$$R \cup R^{-1} = \{(1,2),(2,1),(2,3),(3,2),(3,4),(4,3),(4,5),(5,4),(5,6),(6,5)\}$$

(c) The transitive closure of R is

$$\begin{aligned} R^+ = & \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6), \\ & (3,4),(3,5),(3,6),(4,5),(4,6),(5,6)\} \end{aligned}$$

(d) The reflexive and transitive closure of R is

$$\begin{aligned} R^* &= R^+ \cup Id_X \\ &= \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6), \\ & \quad (3,3),(3,4),(3,5),(3,6),(4,4),(4,5),(4,6),(5,5),(5,6),(6,6)\} \end{aligned}$$

□