▶ **Problem DM-3.5-14** Lex R be the relation on $\{a,b,c,d,e,f,g\}$ defined as

$$R = \{(a,b),(b,c),(c,a),(d,e),(e,f),(f,g)\}$$

Find the smallest integers m and n such that $0 < m < n$ and $R^m = R^n$. Identify the transitive closure of R as well as the transitive, reflexive, and symmetric closure of R.

Solution. Let $X = \{a,b,c,d,e,f,g\}$. Then

$$\begin{aligned}
Id_X = R^0 &= \{(a,a),(b,b),(c,c),(d,d),(e,e),(f,f),(g,g)\}\\
R^{-1} &= \{(a,c),(b,a),(c,b),(e,d),(f,e),(g,f)\}\\
R^2 &= \{(a,c),(b,a),(c,b),(d,f),(e,g)\}\\
R^3 &= \{(a,a),(b,b),(c,c),(d,g)\}\\
R^4 &= \{(a,b),(b,c),(c,a)\}\\
R^5 &= \{(a,c),(b,a),(c,b)\}\\
R^6 &= \{(a,a),(b,b),(c,c)\}\\
R^7 &= \{(a,b),(b,c),(c,a)\} = R^4\\
R^8 &= \{(a,c),(b,a),(c,b)\} = R^5\\
R^9 &= \{(a,a),(b,b),(c,c)\} = R^6
\end{aligned}$$

Thus, $m = 4$ and $n = 7$. In general, $R^{3k-2} = R^4$, $R^{3k-1} = R^5$ and $R^{3k} = R^6$ for $k \geq 2$. Therefore, the transitive closure of R is

$$\begin{aligned}
R^+ \;=\;& \{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c),(d,e),(d,f),(d,g),\\
& (e,f),(e,g),(f,g)\}
\end{aligned}$$

Also, the reflexive and transitive closure of R is

$$\begin{aligned}
R^* &= R^+ \cup Id_X\\
&= \{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c),(d,d),(d,e),(d,f),\\
&\quad (d,g),(e,e),(e,f),(e,g),(f,f),(f,g),(g,g)\}
\end{aligned}$$

Finally, the transitive, reflexive, and symmetric closure of R is

$$\begin{aligned}
R^* \cup (R^*)^{-1} \;=\;& \{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c),(d,d),\\
& (d,e),(d,f),(d,g),(e,e),(e,d),(e,f),(e,g),(f,d),(f,e),(f,f),\\
& (f,g),(g,d),(g,e),(g,f),(g,g)\}
\end{aligned}$$

□