▶ **Problem DM-3.7-12(a)(b)** Let R and S be equivalence relations on a set X.

(a) Show that $R \cap S$ is an equivalence relation.

(b) Show by example that $R \cup S$ need not be an equivalence relation.

Proof. (a) We show that $R \cap S$ is reflexive, symmetric, and transitive as follows.

Reflexive: Let $x \in X$. Since both R and S are reflexive, $(x, x) \in R$ and $(x, x) \in S$. Thus, we have $(x, x) \in R \cap S$ for every $x \in X$. This shows that $R \cap S$ is reflexive.

Symmetric: Let $x, y \in X$ and suppose $(x, y) \in R \cap S$. By the symmetry of R and S, $(x, y) \in R$ and $(x, y) \in S$ imply $(y, x) \in R$ and $(y, x) \in S$. Thus, $(y, x) \in R \cap S$. This shows that $R \cap S$ is symmetric.

Transitive: Let $x, y, z \in X$ and suppose that $(x, y) \in R \cap S$ and $(y, z) \in R \cap S$. We will show that $(x, z) \in R \cap S$. By the transitivity of R and the facts $(x, y) \in R$ and $(y, z) \in R$, we have $(x, z) \in R$. Similarly, by the transitivity of S and the facts $(x, y) \in S$ and $(y, z) \in S$, we also have $(x, z) \in S$. Therefore, $(x, z) \in R \cap S$. This shows that $R \cap S$ is transitive.

(b) Consider $X = \{1, 2, 3\}$ and let R and S be relations on the set X defined as follows:

$$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$$

and

$$S = \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}$$

It is easy to check that both R and S are equivalence relations. However, $R \cup S = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}$ is not transitive since it contains $(1, 2)$ and $(2, 3)$, but $(1, 3) \notin R \cup S$. Therefore, $R \cup S$ is not an equivalence relation.

□