▶ **Problem DM-3.7-16(a)** For $k, n_1, n_2, m_1, m_2 \in \mathbb{N}$, show that if

$$n_1 \equiv n_2 \pmod{k}$$

and

$$m_1 \equiv m_2 \pmod{k}$$

then

$$n_1 + m_1 \equiv n_2 + m_2 \pmod{k}$$

and

$$n_1 \cdot m_1 \equiv n_2 \cdot m_2 \pmod{k}$$

Proof. Suppose that $n_1 \equiv n_2 \pmod{k}$ and $m_1 \equiv m_2 \pmod{k}$. This means that $k|(n_1 - n_2)$ and $k|(m_1 - m_2)$. We now assume that $n_1 - n_2 = k \cdot s$ and $m_1 - m_2 = k \cdot t$ for some integers $s, t \in \mathbb{Z}$.

(1) We first show that $n_1 + m_1 \equiv n_2 + m_2 \pmod{k}$. Obviously, $(n_1 + m_1) - (n_2 + m_2) = (n_1 - n_2) + (m_1 - m_2) = k \cdot s + k \cdot t = k \cdot (s + t)$. Thus, $k|(n_1 + m_1) - (n_2 + m_2)$, which implies that $n_1 + m_1 \equiv n_2 + m_2 \pmod{k}$.

(2) Next, we prove that $n_1 \cdot m_1 \equiv n_2 \cdot m_2 \pmod{k}$ as follows. Since $n_1 - n_2 = k \cdot s$, we have

$$m_1(n_1 - n_2) = m_1 \cdot k \cdot s \tag{1}$$

Similarly, since $m_1 - m_2 = k \cdot t$, we have

$$n_2(m_1 - m_2) = n_2 \cdot k \cdot t \tag{2}$$

By (1) and (2), $n_1 \cdot m_1 - n_2 \cdot m_2 = k(m_1 \cdot s + n_2 \cdot t)$. Thus, $k|(n_1 \cdot m_1 - n_2 \cdot m_2)$, which implies that $n_1 \cdot m_1 \equiv n_2 \cdot m_2 \pmod{k}$.

□