▶ **Problem DM-3.9-12** Show that if R is a linear ordering of a set X, then $R \cup Id_X$ is a partial ordering of X.

Proof. Suppose that R is a linear ordering on X (i.e., R is transitive and satisfies the row of trichotomy). For proving $R \cup Id_X$ is a partial ordering on X, we will show that $R \cup Id_X$ is reflexive, antisymmetric, and transitive.

Reflexive: Since every binary relation containing Id_X is reflexive, $R \cup Id_X$ is reflexive.

Antisymmetric: Suppose $(x, y) \in R \cup Id_X$ and $x \neq y$. Then, $(x, y) \in R$. By the row of trichotomy, we have $(y, x) \notin R$. Clearly, it further implies $(y, x) \notin R \cup Id_X$.

Transitive: Let $x, y, z \in X$. Suppose $(x, y) \in R \cup Id_X$ and $(y, z) \in R \cup Id_X$. Since R is a linear order, the former assumption indicates that either $(x, y) \in R$ or $x = y$, but not both. Similarly, the latter assumption indicates that either $(y, z) \in R$ or $y = z$, but not both. We now consider the following four cases to show that $(x, z) \in R \cup Id_X$:

Case 1: $x = y$ and $y = z$. Then $x = z$ and $(x, z) \in Id_X$. Thus, $(x, z) \in R \cup Id_X$.

Case 3: $x = y$ and $(y, z) \in R$. Then $(x, z) \in R$, and so $(x, z) \in R \cup Id_X$.

Case 2: $(x, y) \in R$ and $y = z$. Then $(x, z) \in R$, and so $(x, z) \in R \cup Id_X$.

Case 4: $(x, y) \in R$ and $(y, z) \in R$. Since R is transitive, we obtain $(x, z) \in R$, and so $(x, z) \in R \cup Id_X$.

Since $R \cup Id_X$ is reflexive, antisymmetric, and transitive, $R \cup Id_X$ is a partial order.

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