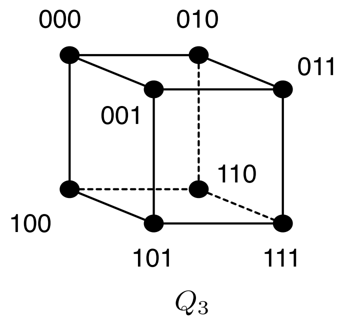
▶ **Problem DM-6.6-12** Prove that Q_n where n is some integral power of 2 has 2^n vertices and $n \cdot 2^{n-1}$ edges.

Solution. Let $V(G)$ and $E(G)$ denote the vertex set and edge set of a graph G. The proof is by induction on integer n. For the base case, we are easy to check from the figure that $|V(Q_1)| = 2$, $|E(Q_1)| = 1$; $|V(Q_2)| = 4$, $|E(Q_2)| = 4$; and $|V(Q_3)| = 8$, $|E(Q_3)| = 12$.



Suppose that $|V(Q_{n-1})| = 2^{n-1}$ and $|E(Q_{n-1})| = (n-1) \cdot 2^{n-2}$. We now consider Q_n as follows.

Recall that an n-cube Q_n can be constructed from two copies of $(n-1)$-cube such that every edge connects this two copies if and only if the binary representations for the two processors differ in exactly one position. Thus, the number of edge connecting this two copies of Q_{n-1} is totally $|V(Q_{n-1})| = 2^{n-1}$. Consequently,

$$|V(Q_n)| = 2|V(Q_{n-1})| = 2 \cdot 2^{n-1} = 2^n$$

and

$$|E(Q_n)| = 2|E(Q_{n-1})| + |V(Q_{n-1})| = 2 \cdot (n-1) \cdot 2^{n-2} + 2^{n-1} = n \cdot 2^{n-1}.$$

□