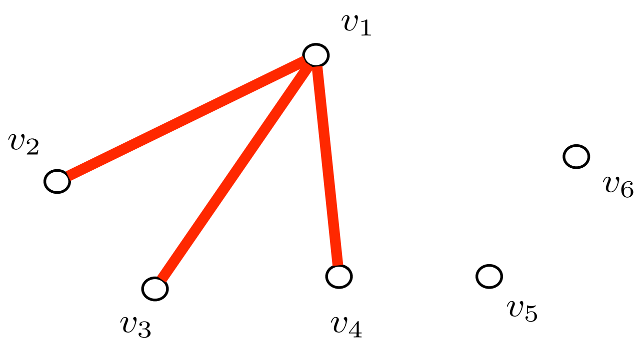
▶ **Problem DM-6.6-14** Prove that for any graph G on six vertices, either G or $\overline{G}$ contains a triangle (i.e., a cycle on three vertices).

Proof. Let G be any graph on six vertices labeled by $v_1, v_2, \ldots, v_6$. It is clear that $G \cup \overline{G} = K_6$ (a complete graph on six vertices). Suppose that G has each edge colored by red and $\overline{G}$ has each edge colored by blue. Then, the union of G and $\overline{G}$ forms a red-blue coloring of K_6. We now show that we can find 3 vertices in K_6 such that the 3 edges joining them are the same color.

Consider some vertex v_1 of K_6. Since v_1 is incident with five edges, it follows by the Pigeonhole Principle that at least three of these five edges are colored the same, say red. Suppose that $(v_1, v_2), (v_1, v_3), (v_1, v_4)$ are red edges, as shown in the following Figure.



If any of the edges $(v_2, v_3), (v_2, v_4)$ ans (v_3, v_4) is colored red, then we have a red K_3; otherwise, all of these edges are colored blue, and a blue K_3 is formed.

$\square$