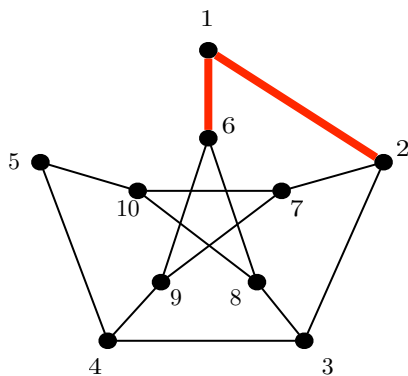▶ **Problem DM-6.6-26** Complete the proof of Case ii of Example 6 in Section 6.3.1.

Proof.

    Case ii: Suppose $(1, 2)$ and $(1, 6)$ are contained in the Hamiltonian cycle. It follows by condition 4 that $(1, 5)$ can be removed from further consideration. We must finish a Hamiltonian cycle in the following graph.

Since vertex 5 has degree two, the edges $(5, 4)$ and $(5, 10)$ must be contained in every Hamiltonian cycle by condition 2. Thus, we further consider the following graph.
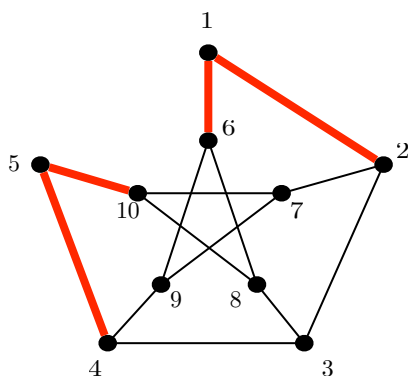
Now, the second edges incident to vertex 2 and 4 must be one of the following four pairs of edges:

  (a) $(2, 7)$ and $(4, 9)$

  (b) $(2, 7)$ and $(4, 3)$ or, symmetrically, $(4, 9)$ and $(2, 3)$

  (c) $(2, 3)$ and $(4, 3)$

Case (a): If edges $(2, 7)$ and $(4, 9)$ are chosen, the edges $(2, 3)$ and $(4, 3)$ can be deleted by condition 4. In this case, vertex 3 has degree one in the remaining graph (see below), which violates the condition 1.

Case (b): The choice of $(2, 7)$ and $(4, 3)$ will allow edges $(2, 3)$ and $(4, 9)$ to be deleted by condition 4. The edges $(6, 9)$, $(7, 9)$ and $(3, 8)$ must now be included in the Hamiltonian cycle by condition 2 (see below). In this case, the choice of an edge $(8, 10)$ will lead to a violation of condition 5.

Case (c): The choice of $(2, 3)$ and $(4, 3)$ will allow edges $(2, 7)$, $(4, 9)$ and $(3, 8)$ to be deleted by condition 4 (see below). Since each of the vertices 7, 8 and 9 has degree 2 in the remaining graph, all the incident edges must be included in the Hamiltonian cycle by condition 2. Therefore, it leads to a violation of condition 3.

□