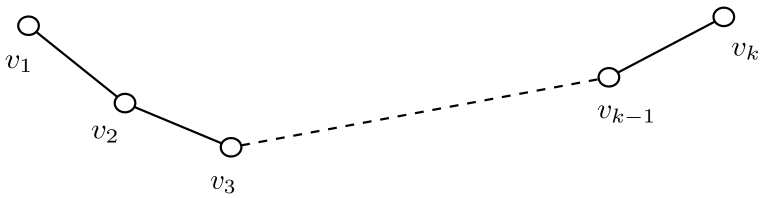
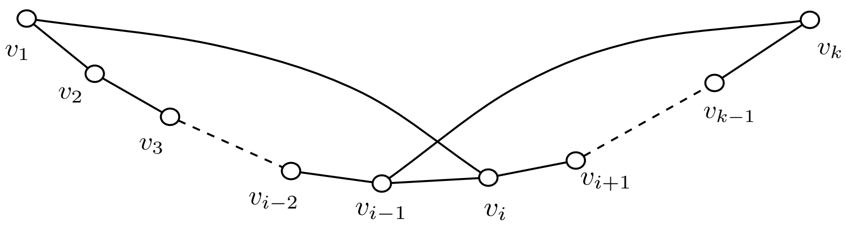
▶ **Problem DM-6.6-40** Let $G = (V, E)$ be a graph with $|V| \geq 3$. Prove that if the degree of each vertex in G is at least $|V|/2$, then G is Hamiltonian.

Proof. Let G be a graph with n vertices (i.e., $n = |V|$). If $n = 3$, then the condition on G implies that G is isomorphic to K_3, and hence G is hamiltonian. We may assume, therefore, that $n \geq 4$.

Let $P = (v_1, v_2, \ldots, v_k)$ be a longest path in G (see the following figure). Then every neighbor of v_1 and every neighbor of v_k is on P; otherwise, there would be a longer path than P in G. Consequently, $k \geq 1 + \frac{n}{2}$.



In the following, we show that there must be some vertex v_i, where $2 \leq i \leq k$, such that v_1 is adjacent to v_i, and v_k is adjacent to v_{i-1}. If this were not the case, then, whenever v_1 is adjacent to a vertex v_i, the vertex v_k is *not* adjacent to v_{i-1}. Since at least $\frac{n}{2}$ vertices on P are adjacent to v_1, at least $\frac{n}{2}$ of the $n-1$ vertices different from v_k on P are not adjacent to v_k. Hence, $deg(v_k) \leq (n-1) - \frac{n}{2} < \frac{n}{2}$, which contradicts the fact that $deg(v_k) \geq \frac{n}{2}$. Therefore, as we claimed, there must be a vertex v_i adjacent to v_1, and v_{i-1} is adjacent to v_k (see the following figure).



We now see that G has a cycle $C = (v_1, v_i, v_{i+1}, \ldots, v_{k-1}, v_k, v_{i-1}, v_{i-2}, \ldots, v_2, v_1)$ that contains all the vertices of P. If C contains all the vertices of G (i.e., $k = n$), then C is a Hamiltonian cycle, and the proof is complete. Otherwise, there is some vertex u of G that is not on C. By hypothesis, $deg(u) \geq \frac{n}{2}$. Since P contains at least $1 + \frac{n}{2}$ vertices, there are fewer than $\frac{n}{2}$ vertices not on C. This implies that u must be adjacent to a vertex v that lies on C. However, the edge (u, v) together with the cycle C contain a path whose length is greater than that of P, which is impossible. Thus, C contains all vertices of G, and G is Hamiltonian. $\square$