▶ **Problem DM-6.9-4** Let G be a graph. Prove that if G is disconnected, then $\overline{G}$ is connected.

Proof. Suppose that $G = (V, E)$ is a disconnected graph. Clearly, G must contain at least two vertices. Let $x, y \in V$ belong to different connected components of G. Then no vertex in the set $V - \{x, y\}$ is adjacent to both x and y in G. We now consider $\overline{G} = (V, E')$. First, it is obvious that $(x, y) \in E'$. Second, any other vertex z is adjacent to at least one of x and y in $\overline{G}$ (To see this, we suppose to the contrary that z is not adjacent to both x and y in $\overline{G}$. This leads to a contradiction that $z \in V - \{x, y\}$ is adjacent to both x and y in G.)

In the following, we will show that any two vertex z and w in $\overline{G}$ are connected by a path, and thus $\overline{G}$ is connected. Obviously, if both z and w are adjacent to x (respectively, to y) in $\overline{G}$, then z-x-w (respectively, z-y-w) is a path connecting z and w in $\overline{G}$. Thus, we only need to consider that neither x nor y is adjacent to both z and w in $\overline{G}$. In this case, form the previous argument we have shown that either $(z, x), (w, y) \in E'$ or $(z, y), (w, x) \in E'$. The former case implies that z-x-y-w is a path connecting z and w in $\overline{G}$, and the latter case implies that z-y-x-w is a path connecting z and w in $\overline{G}$. □