▶ **Problem 2.4-06** For natural numbers a and b, define $a \sim b$ if and only if $a^2 + b$ is even. Prove that $\sim$ defines an equivalence relation on $\mathbf{N}$ and find the quotient set determined by $\sim$.

Proof. (a) To show that $\sim$ is an equivalence on $\mathbf{N}$, we verify the reflexive, symmetric, and transitive as follows.

(i). **Reflexive** : For any $a \in \mathbf{N}$, $a \sim a$ because $a^2 + a = a(a+1)$ is even, as the product of consecutive natural numbers.

(ii). **Symmetric** : For any $a, b \in \mathbf{N}$, if $a \sim b$ then $a^2 + b$ is even. It follows that $b^2 + a$ is even since $(a^2 + b) + (b^2 + a) = a(a+1) + b(b+1)$ and all the terms $a^2 + b$, $a(a+1)$, and $b(b+1)$ are even. Thus, $b \sim a$.

(iii). **Transitive** : For any $a, b, c \in S$, if $a \sim b$ and $b \sim c$, then $a^2 + b$ and $b^2 + c$ are both even, so $(a^2 + b) + (b^2 + c)$ is even. In other word, $(a^2 + c) + (b^2 + b)$ is even. Since $b^2 + b = b(b+1)$ is even, $a^2 + c$ is even too. Thus, $a \sim c$.

(b) The quotient set is the set of equivalence classes. Now

$$\overline{a} = \{x \in \mathbf{N} \,|\, x \sim a\} = \{x \in \mathbf{N} \,|\, x^2 + a \text{ is even}\} = \begin{cases} 2\mathbf{Z} & \text{if } a \text{ is even} \\ 2\mathbf{Z} + 1 & \text{if } a \text{ is odd.} \end{cases}$$

Thus, $\mathbf{N}/\sim = \{2\mathbf{Z}, 2\mathbf{Z} + 1\}$. □