▶ **Problem 2.4-09** Define $\sim$ on $\mathbf{Z}$ by $a \sim b$ if and only if $3a + b$ is a multiple of 4.

(a) Prove that $\sim$ defines an equivalence.

(b) Find the equivalence class of 0.

(c) Find the equivalence class of 2.

(d) Make a guess about the quotient set.

Proof. (a) To show that $\sim$ is an equivalence on $\mathbf{Z}$, we verify the reflexive, symmetric, and transitive as follows.

(i). **Reflexive** : For any $a \in \mathbf{Z}$, $a \sim a$ because $3a + a = 4a$ is a multiple of 4.

(ii). **Symmetric** : For any $a, b \in \mathbf{Z}$, if $a \sim b$ then $3a + b = 4n$ for some integer $n \in \mathbf{Z}$. Since $a + b - n$ is an integer, we have $3b + a = (4a + 4b) - (3a + b) = (4a + 4b) - 4n = 4(a + b - n)$, which is a multiple of 4. Thus $b \sim a$.

(iii). **Transitive** : For any $a, b, c \in \mathbf{Z}$, if $a \sim b$ and $b \sim c$ then $3a+b = 4n$ and $3b+c = 4m$ for some integers $n, m \in \mathbf{Z}$. Thus, $(3a + b) + (3b + c) = 4(n + m)$. Since $n + m - b$ is an integer, $3a + c = (3a + b) + (3b + c) - 4b = 4(n + m) - 4b = 4(n + m - b)$ is a multiple of 4. Therefore, $a \sim c$.

(b) $\overline{0} = \{a \in \mathbf{Z} \,|\, 0 \sim a\} = \{a \in \mathbf{Z} \,|\, 3 \cdot 0 + a = 4k \text{ for some } k \in \mathbf{Z}\} = \{4k \,|\, k \in \mathbf{Z}\} = 4\mathbf{Z}$.

(c) $\overline{2} = \{a \in \mathbf{Z} \,|\, 2 \sim a\} = \{a \in \mathbf{Z} \,|\, 3 \cdot 2 + a = 4k \text{ for some } k \in \mathbf{Z}\} = \{4k - 6 \,|\, k \in \mathbf{Z}\} = \{4k + 2 \,|\, k \in \mathbf{Z}\} = 4\mathbf{Z} + 2$.

(d) The quotient set is $\mathbf{Z}/\sim = \{4\mathbf{Z}, 4\mathbf{Z} + 1, 4\mathbf{Z} + 2, 4\mathbf{Z} + 3\}$. □