▶ **Problem 2.5-10** (a) Let $A = \mathbf{Z}^2$ and for $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ in A, define $\mathbf{a} \preceq \mathbf{b}$ if and only if $a_1 \le b_1$ and $a_1 + a_2 \le b_1 + b_2$. Prove that $\preceq$ is a partial order on A. Is this partial order a total order? Justify your answer with a proof or a counterexample.

(b) Generalize the result of part (a) by defining a partial order on the set $\mathbf{Z}^n$ of n-tuples of integers. (No proof is required.)

Proof. (a) To show that $\preceq$ is a partial order, we verify the reflexive, antisymmetric, and transitive as follows.

(i). **Reflexive** : For any $\mathbf{a} = (a_1, a_2) \in A$, $\mathbf{a} \preceq \mathbf{a}$ because $a_1 \le a_1$ and $a_1 + a_2 \le a_1 + a_2$.

(ii). **Antisymmetric** : If $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ are elements in A with $\mathbf{a} \preceq \mathbf{b}$ and $\mathbf{b} \preceq \mathbf{a}$, then $a_1 \le b_1$, $a_1 + a_2 \le b_1 + b_2$ and $b_1 \le a_1$, $b_1 + b_2 \le a_1 + a_2$. Since $a_1 \le b_1$ and $b_1 \le a_1$, it implies $a_1 = b_1$. Also, since $a_1 + a_2 \le b_1 + b_2$ and $a_1 = b_1$, we have $a_2 \le b_2$. Similarly, $b_2 \le a_2$, so $a_2 = b_2$. Thus, $\mathbf{a} = \mathbf{b}$.

(iii). **Transitive** : If $\mathbf{a} = (a_1, a_2)$, $\mathbf{b} = (b_1, b_2)$ and $\mathbf{c} = (c_1, c_2)$ are elements in A with $\mathbf{a} \preceq \mathbf{b}$ and $\mathbf{b} \preceq \mathbf{c}$, then $a_1 \le b_1$, $a_1 + a_2 \le b_1 + b_2$ and $b_1 \le c_1$, $b_1 + b_2 \le c_1 + c_2$. Since $a_1 \le b_1$ and $b_1 \le c_1$, it implies $a_1 \le c_1$. Also, since $a_1 + a_2 \le b_1 + b_2$ and $b_1 + b_2 \le c_1 + c_2$, we have $a_1 + a_2 \le c_1 + c_2$. Thus, $\mathbf{a} \preceq \mathbf{c}$.

This parial order is not a total order. For example, $\mathbf{a} = (1, 6)$ and $\mathbf{b} = (2, 4)$ are not comparable since $a_1 = 1 < b_1 = 2$ but $a_1 + a_2 = 1 + 6 = 7 > b_1 + b_2 = 2 + 4 = 6$.

(b) Let $A = \mathbf{Z}^n$ and for $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ in A, define $\mathbf{a} \preceq \mathbf{b}$ if and only if $\sum_{i=1}^{k} a_i \le \sum_{i=1}^{k} b_i$ for every $k = 1, 2, \ldots, n$. Then $\preceq$ is a partial order on A (By the description of the problem, we omit the proof). $\square$