▶ **Problem 3.1-19(b)(c)(d)** Find the domain and range of each of the given functions of a real variable. In each case, determine whether the function is one-to-one and whether it is onto.

(b) $g : \mathbf{R} \to \mathbf{R}$ defined by $g(x) = x|x|$.

(c) $\beta : (\frac{4}{3}, \infty) \to \mathbf{R}$ defined by $\beta(x) = \log_2(3x - 4)$.

(d) $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 2^{x-1} + 3$.

Solution. (b) The domain of g is $\mathbf{R}$ and so is the range. For the latter, note that

$$g(x) = \begin{cases} x^2 & \text{if } x \geqslant 0 \\ -x^2 & \text{if } x < 0 \end{cases}$$

so that if $y \geqslant 0$, then $y = g(\sqrt{y})$; while if $y < 0$, then $y = g(-\sqrt{-y})$. Since rng $g = \mathbf{R}$, g is onto. To see that g is one-to-one, we suppose $g(x_1) = g(x_2)$ and consider the following cases:

Case 1: Both x_1 and x_2 are nonnegative.
In this case, $x_1^2 = x_2^2$, so $\sqrt{x_1^2} = \sqrt{x_2^2}$, and hence $|x_1| = |x_2|$. Thus $x_1 = x_2$.

Case 2: Both x_1 and x_2 are negative.
In this case, $-x_1^2 = -x_2^2$, so $x_1^2 = x_2^2$, $\sqrt{x_1^2} = \sqrt{x_2^2}$, and hence $|x_1| = |x_2|$, $-x_1 = -x_2$. Thus $x_1 = x_2$.

Case 3: One of x_1 and x_2 is nonnegative and the other is negative.
Here we may assume that x_1 is nonnegative, so $x_1^2 = -x_2^2$. Since the left side is at least 0, and the right side is less then 0, the equation cannot be true.

In all cases $x_1 = x_2$, so g is one-to-one.

(c) The domain of β is $(\frac{4}{3}, \infty)$ as given. The range of β is $\mathbf{R}$ because for any $y \in \mathbf{R}$, $y = \beta(x)$ for $x = \frac{1}{3}(2^y + 4)$. Since rng $\beta = \mathbf{R}$, β is onto. To show that β is one-to-one, we see that if $\beta(x_1) = \beta(x_2)$, then $\log_2(3x_1 - 4) = \log_2(3x_2 - 4)$. So, $2^{\log_2(3x_1-4)} = 2^{\log_2(3x_2-4)}$, $3x_1 - 4 = 3x_2 - 4$ and $x_1 = x_2$.

(d) The domain of f is $\mathbf{R}$ as given. To fine the range, we recall that $2^t > 0$ for all t. Thus $f(x) > 3$ for all x. In fact, rng $f = (3, \infty)$ since, for any $y > 3$, $y = f(x)$ for $x = 1 + \log_2(y - 3)$. Since rng $f \neq \mathbf{R}$, f is not onto. It is one-to-one, however, for if $f(x_1) = f(x_2)$, then $2^{x_1-1} + 3 = 2^{x_2-1} + 3$, so $2^{x_1-1} = 2^{x_2-1}$. Taking $\log_2$ of each side, we have $x_1 - 1 = x_2 - 1$. Thus $x_1 = x_2$. $\square$