▶ **Problem 3-1-26** Let S be a set containing the number 5. Let $A = \{f : S \to S\}$ be the set of all functions $S \to S$. For $f, g \in A$, define $f \sim g$ if $f(5) = g(5)$.

(a) Prove that $\sim$ defines an equivalence relation on A.

(b) Find the equivalence class of $f = \{(5, a), (a, b), (b, b)\}$ in the case $S = \{5, a, b\}$.

Proof. (a) To show that $\sim$ is an equivalence on A, we verify the reflexive, symmetric, and transitive as follows.

(i). **Reflexive** : If $f \in A$, then $f(5) = f(5)$, so $f \sim f$.

(ii). **Symmetric** : For any $f, g \in A$, if $f \sim g$ then $f(5) = g(5)$, so $g(5) = f(5)$. Thus $g \sim f$.

(iii). **Transitive** : For any $f, g, h \in \mathbf{Z}$, if $f \sim g$ and $g \sim h$ then $f(5) = g(5)$ and $g(5) = h(5)$. Thus, $f(5) = g(5) = h(5)$ and $f \sim h$.

(b) The equivalence class of f is $\overline{f} = \{g : S \to S \,|\, g(5) = f(5) = a\}$. Since there are three possible images for each of $g(a)$ and $g(b)$, there are altogether 9 functions in $\overline{f}$, namely,

$$\begin{aligned}
&f_1 = \{(5, a), (a, 5), (b, 5)\}, \;\; f_2 = \{(5, a), (a, 5), (b, a)\}, \;\; f_3 = \{(5, a), (a, 5), (b, b)\}, \\
&f_4 = \{(5, a), (a, a), (b, 5)\}, \;\; f_5 = \{(5, a), (a, a), (b, a)\}, \;\; f_6 = \{(5, a), (a, a), (b, b)\}, \\
&f_7 = \{(5, a), (a, b), (b, 5)\}, \;\; f_8 = \{(5, a), (a, b), (b, a)\}, \;\; f_9 = \{(5, a), (a, b), (b, b)\}.
\end{aligned}$$

□