▶ **Problem 3.2-7(c)(d)** Show that each of the following functions $f : A \to \mathbf{R}$ is one-to-one. Find the range of each function and a suitable inverse.

(c) $A = \{x \in \mathbf{R} \,|\, x \neq -\frac{1}{2}\}$, $f(x) = \frac{3x}{2x+1}$.

(d) $A = \{x \in \mathbf{R} \,|\, x \neq -3\}$, $f(x) = \frac{x-3}{x+3}$.

Solution. (c) Suppose $f(x_1) = f(x_2)$. Then $\frac{3x_1}{2x_1+1} = \frac{3x_2}{2x_2+1}$, so $6x_1x_2 + 3x_1 = 6x_1x_2 + 3x_2$ and $x_1 = x_2$. Thus f is one-to-one. Now

$$\begin{aligned}
y \in \text{ rng } f \quad &\leftrightarrow \quad y = f(x) \text{ for some } x \in A \\
&\leftrightarrow \quad \text{there is an } x \in A \text{ such that } y = \frac{3x}{2x+1} \\
&\leftrightarrow \quad \text{there is an } x \in A \text{ such that } 2xy + y = 3x \\
&\leftrightarrow \quad \text{there is an } x \in A \text{ such that } x(2y-3) = -y \\
&\leftrightarrow \quad y \neq \frac{3}{2}
\end{aligned}$$

Thus, rng $f = B = \{y \in \mathbf{R} \,|\, y \neq \frac{3}{2}\}$ and f has an inverse $f^{-1} : B \to A$. To find a formula $f^{-1}(x)$, let $y = f^{-1}(x)$ where $x \in B$. Then $x = f(y) = \frac{3y}{2y+1}$, so $2xy + x = 3y$ and $y(2x-3) = -x$. Since $x \in B$, we know $2x - 3 \neq 0$. Thus, $f^{-1}(x) = y = \frac{-x}{2x-3}$.

(d) Suppose $f(x_1) = f(x_2)$. Then $\frac{x_1-3}{x_1+3} = \frac{x_2-3}{x_2+3}$, so $x_1x_2 + 3x_1 - 3x_2 - 9 = x_1x_2 - 3x_1 + 3x_2 - 9$, $6x_1 = 6x_2$ and $x_1 = x_2$. Thus f is one-to-one. Now

$$\begin{aligned}
y \in \text{ rng } f \quad &\leftrightarrow \quad y = f(x) \text{ for some } x \in A \\
&\leftrightarrow \quad \text{there is an } x \in A \text{ such that } y = \frac{x-3}{x+3} \\
&\leftrightarrow \quad \text{there is an } x \in A \text{ such that } y(x+3) = x-3 \\
&\leftrightarrow \quad \text{there is an } x \in A \text{ such that } x - yx = 3y + 3 \\
&\leftrightarrow \quad \text{there is an } x \in A \text{ such that } x(1-y) = 3(y+1) \\
&\leftrightarrow \quad y \neq 1
\end{aligned}$$

Thus, rng $f = B = \{y \in \mathbf{R} \,|\, y \neq 1\}$ and f has an inverse $f^{-1} : B \to A$. To find a formula $f^{-1}(x)$, let $y = f^{-1}(x)$ where $x \in B$. Then $x = f(y) = \frac{y-3}{y+3}$, so $x(y+3) = y - 3$ and $y(1-x) = 3(x+1)$. Since $x \in B$, we know $x \neq 1$. Thus, $f^{-1}(x) = y = \frac{3(1+x)}{1-x}$.

□