▶ **Problem 3-2-17** Let A denote the set $\mathbf{R} \setminus \{0, 1\}$. Let ι denote the identity function on A and define the functions $f, g, h, s, r : A \to A$ by

$$f(x) = 1 - \frac{1}{x}, \quad g(x) = \frac{1}{1-x}, \quad h(x) = \frac{1}{x}, \quad r(x) = \frac{x}{x-1}, \quad s(x) = 1 - x.$$

(a) Show that $f \circ g = \iota$ and $g \circ r = s$. Complete the table, thereby showing that the composition of any two of the given functions is one of the given five, or the identity.

$\circ$	ι	f	g	h	r	s
ι						
f			ι			
g					s	
h						
r						
s						

(b) Which of the given six functions have inverses? Find (and identify) any inverses that exist.

Proof. (a) $f \circ g(x) = f(g(x)) = f(\frac{1}{1-x}) = 1 - \frac{1}{\frac{1}{1-x}} = 1 - (1 - x) = \iota(x)$.

$g \circ r(x) = g(r(x)) = g(\frac{x}{x-1}) = \frac{1}{1-\frac{x}{x-1}} = \frac{x-1}{(x-1)-x} = \frac{x-1}{-1} = 1 - x = s(x)$

$\circ$	ι	f	g	h	r	s
ι	ι	f	g	h	r	s
f	f	g	ι	s	h	r
g	g	ι	f	r	s	h
h	h	r	s	ι	f	g
r	r	s	h	g	ι	f
s	s	h	r	f	g	ι

(b) All these function have inverse. See the following table.

Function	ι	f	g	h	r	s
Inverse	ι	g	f	h	r	s

□