▶ **Problem 3-3-14** Let a and b be real numbers with $a < b$. Show that the set $(0, \infty)$ of positive real numbers has the same cardinality as the open interval (a, b).

Proof. As in Problem 27 of the textbook (Page 89), we saw that the function $g : (0, 1) \to (0, \infty)$ defined by $g(x) = \frac{1}{x} - 1$ is one-to-one correspondence.

Now, we define the function $f : (a, b) \to (0, 1)$ by given $f(x) = \frac{x-a}{b-a}$. Note that f is onto, because if $y = f(x)$ then $x = (b - a)y + a$, and if $0 < y < 1$ then $(b - a) \cdot 0 + a < (b - a)y + a < (b - a) \cdot 1 + a$; that is $a < x < b$. Also, f in one-to-one because

$$f(x_1) = f(x_2) \ \to \ \frac{x_1 - a}{b - a} = \frac{x_2 - a}{b - a} \ \to \ x_1 - a = x_2 - a \ \to \ x_1 = x_2.$$

Thus f is a one-to-one correspondence between (a, b) and $(0, 1)$.

Therefore, the function $g \circ f : (a, b) \to (0, \infty)$ given by

$$(g \circ f)(x) = \frac{b - a}{x - a} - 1$$

is a bijection, so the sets $(0, \infty)$ and (a, b) have the same cardinality. □