▶ **Problem 3-3-29** Let S be the set of all real numbers in the interval $(0,1)$ whose decimal expansions are infinite and contain only 3 and 4, for example, $0.343434\ldots$ and $0.333\ldots$, but not $0.34 = 0.34000\ldots$. Prove that S is uncountable.

Proof. Assume, to the contrary, that S is countable and, as in Problem 31 of the textbook (Page 92), we write each of its elements in a list as

$$\begin{aligned} a_1 &= 0.a_{11}a_{12}a_{13}a_{14}\cdots \\ a_2 &= 0.a_{21}a_{22}a_{23}a_{24}\cdots \\ a_3 &= 0.a_{31}a_{32}a_{33}a_{34}\cdots \\ &\vdots \end{aligned}$$

where each a_{ij} is 3 or 4. Define the sequence $b_1, b_2, b_3, \ldots$ by $b_i = \begin{cases} 4 & \text{if } a_{ii} = 3 \\ 3 & \text{if } a_{ii} = 4. \end{cases}$ Then $b = 0.b_1b_2b_3\ldots$ is in S, yet it is different from each a_i because $b_i \neq a_{ii}$ for each i. This contradiction gives the result. □