▶ **Problem 4.5-21(c)** Find a positive integer x such that $ab \equiv x$ modulo a suitable integer. Assuming that $ab < 50000$, find ab itself, if possible, and explain your reasoning.
$a \equiv 1,\ b \equiv 2 \pmod{8}$
$a \equiv 8,\ b \equiv 1 \pmod{27}$
$a \equiv 55,\ b \equiv 82 \pmod{125}$

Solution.

$ab \equiv 1 \cdot 2 \equiv 2 \pmod{8}$
$ab \equiv 8 \cdot 1 \equiv 8 \pmod{27}$
$ab \equiv 55 \cdot 82 \equiv 10 \pmod{125}$

We first solve $ab \equiv 2 \pmod{8}$ and $ab \equiv 8 \pmod{27}$
Since $1 = 3 \cdot 27 + (-10) \cdot 8$, we obtain
$ab \equiv 2 \cdot 3 \cdot 27 + 8 \cdot (-10) \cdot 8 = -478 \equiv 170 \pmod{216}$.

Then, we solve $ab \equiv 10 \pmod{125}$ and $ab \equiv 170 \pmod{216}$.
Since $1 = (-19) \cdot 125 + 11 \cdot 216$, we obtain
$ab \equiv 170 \cdot (-19) \cdot 125 + 10 \cdot 11 \cdot 216 = -379990 \equiv 25010 \pmod{27000}$.
Since $ab < 50000$, $x = 25010$ is the unique solution. □