▶ **Problem 9.2-35**

Can there exist a graph with 13 vertices, 31 edges, 3 vertices of degree 1, and 7 vertices of degree 4? Explain.

Solution. No, there cannot. Let v_1, v_2 and v_3 be the remaining three vertices. The sum of the degrees for the ten vertices given is $3 \cdot 1 + 7 \cdot 4 = 31$, but the total degrees of the graph equals to $2|E| = 62$. It follows that $\sum_{i=1}^{3} \deg\ v_i = 31$. Since the graph has just 13 vertices, we have $\deg\ v_i \leqslant 12$ for every $i = 1, 2, 3$, and at least one of v_1, v_2, v_3 has degree at least $\lceil \frac{31}{3} \rceil = 11$. We consider the following two cases.

Case 1: There is a vertex of degree 12.
Without loss of generality we assume that $\deg\ v_1 = 12$. In this case, v_1 is adjacent to all other vertices, in particular to all the three vertices of degree one. Since these three vertices of degree 1 cannot be joined to any other vertex, the degree of any other vertex is at most nine. Thus, $\sum_{i=1}^{3} \deg\ v_i \leqslant 12 + 9 + 9 = 30 < 31$, which is a contradiction. So this case is impossible.

Case 2: There is no vertex of degree 12.
Without loss of generality we assume that $\deg\ v_1 = 11$. Then v_1 must be adjacent to at least two of the vertices of degree one. Since these vertices of degree 1 cannot be joined to any other vertex, $\deg\ v_2 \leqslant 10$ and $\deg\ v_3 \leqslant 10$. If one of v_2 and v_3 had degree 10, it would be joined to the third vertex of degree one, and the final vertex would be of degree at most 9. This is impossible because $\sum_{i=1}^{3} \deg\ v_i \leqslant 11 + 10 + 9 = 30 < 31$. Otherwise, $\deg\ v_2 \leqslant 9$ and $\deg\ v_3 \leqslant 9$. Thus, $\sum_{i=1}^{3} \deg\ v_i \leqslant 11 + 9 + 9 = 29 < 31$, again a contradiction. □