

► **Problem 10.1-13**

- (a) Find a necessary and sufficient condition on natural numbers m and n in order for $K_{m,n}$ to be Eulerian. Prove your answer.
- (b) Find a necessary and sufficient condition on natural numbers m and n in order for $K_{m,n}$ to have an Eulerian trail. Assume $m \leq n$. Prove your answer.

Solution. (a) Since $K_{m,n}$ is connected, we seek only necessary and sufficient conditions for every vertex to be even. In $K_{m,n}$, every vertex in the bipartition set with m vertices has degree n and every vertex in the other bipartition set has degree m . Hence, $K_{m,n}$ is Eulerian if and only if m and n are both even.

(b) $K_{m,n}$ with $m \leq n$ has Eulerian trail if and only if the following conditions hold: (i) both $m, n > 0$ are even numbers; (ii) $m = 1$ and $n = 2$; or (iii) $m = 2$ and n is odd. Obviously, every Eulerian graph has a Eulerian trail. This shows that the condition (i) holds. Also, the condition (ii) is easy to verify. By Theorem 10.1.5, complete bipartite graphs with the specified conditions on m and n have Eulerian trail. Since $K_{m,n}$ has n vertices of degree m and m vertices of degree n , if $m > 2$, then each bipartition set contains more than two vertices (since $n \geq m$) so it is impossible for there to exist precisely two vertices of odd degree. □