▶ Problem 10.1-24

Let G be a graph with n vertices and m edges, where $m > \frac{1}{2}(n-1)(n-2)$.

(a) Show that G does not have a vertices of degree 0.

(b) Show that G is connected.

Proof. (a) If G has a vertex of degree 0, then G would have the most edges in the case that each pair of the remaining $n-1$ vertices are joined by an edge. The graph have at most $\binom{n-1}{2} = \frac{(n-1)(n-2)}{2} < m$ edges.

(b) Suppose to the contrary that G is not a connected graph. Then, we can find two vertices u and v in G which are not joined to each other by a walk. In particular, u and v are not adjacent to any common vertex and they are not adjacent to each other. We conclude that $\deg\ u + \deg\ v \leqslant n-2$, and the remaining $n-2$ vertices each have degree at most $n-2$. Thus, the sum of the degrees of all vertices is at most $(n-2)+(n-2)(n-2) = (n-1)(n-2)$. That is,

$$m = \frac{1}{2}\sum_{v \in V} \deg\ v \leqslant \frac{(n-1)(n-2)}{2}.$$

This is a contradiction. □