▶ **Problem 10.2-16** [Ore's Theorem]

Suppose G is a graph with $n \geqslant 3$ vertices and that the sum of the degrees of any two nonadjacent vertices is at least n. Prove that G is Hamiltonian by starting with a path $P : v_1 v_2 \cdots v_t$ of greatest length, as in the proof of Dirac's Theorem, and then considering separately the cases where

(a) v_1 and v_t are adjacent, and

(b) v_1 and v_t are not adjacent

Proof. (b) Suppose that v_1 and v_t are not adjacent. By hypothesis, $deg\, v_1 + deg\, v_t \geqslant n$. As in the proof of Dirac's Theorem, we have known that every vertex adjacent to v_1 (respectively, v_t) is in P. Now, we claim that there is a pair of vertices v_k, v_{k+1} in P, $1 \leqslant k < t$, such that v_1 is adjacent to v_{k+1} and v_t is adjacent to v_k. Suppose that it is not the case. Then, every vertex of P adjacent to v_1 would imply that its predecessor in P is not adjacent to v_t. Since every vertex adjacent to v_1 is contained in P, at least $1 + deg\, v_1$ vertices in P are not adjacent to v_t. Hence, at most $n - 1 - deg\, v_1$ vertices are adjacent to v_k in G. Therefore, $deg\, v_1 + deg\, v_t \leqslant n - 1$, a contradiction. It follows that $C : v_1 v_{k+1} v_{k+2} \cdots v_t v_k v_{k-1} \cdots v_1$ is a cycle, and so there is a path of length t beginning at v_k and ending at v_t. The argument given in (a) for the case where v_1 and v_t are adjacent now shows that C is Hamiltonian. □