▶ **Problem 10.2-17**

Suppose G is a graph with $n \geqslant 3$ vertices and at least $\binom{n-1}{2} + 2$ edges. Show that G is Hamiltonian.

Proof. By Ore's Theorem (see Problem 10.2.16), it is sufficient to show that the sum of the degrees of any two nonadjacent vertices is at least n. So, let u and v be any two nonadjacent vertices in the graph $G = (V, E)$. If we delete u and v (and all edges incident with u or v), we are left with a graph G' with $n - 2$ vertices and $|E| - \deg\, u - \deg\, v$ edges. Clearly, G' has at most $\binom{n-2}{2}$ edges. Since

$$\binom{n-1}{2} + 2 - \deg\, u - \deg\, v \leqslant |E| - \deg\, u - \deg\, v \leqslant \binom{n-2}{2},$$

it implies

$$\begin{aligned} \deg\, u + \deg\, v &\geqslant \binom{n-1}{2} - \binom{n-2}{2} + 2 \\ &= \frac{(n-1)(n-2)}{2} - \frac{(n-2)(n-3)}{2} + 2 \\ &= \frac{(n-2)\Big[(n-1) - (n-3)\Big]}{2} + 2 \\ &= (n-2) + 2 \\ &= n \end{aligned}$$

and the result follows. □