▶ **Problem 11.4-09**

Complete the proof of Theorem 11.4.4 as follows.

(a) Prove that the scores in a transitive tournament of n vertices are all different. Thus, $(2) \to (3)$.

(b) Suppose every player in a tournament T has a different score. Show that T has a a unique Hamiltonian path. Thus, $(3) \to (1)$. [*Hint* : PAUSEs 9 and 11]

Proof. (a) Suppose that v and w are two players in a transitive tournament T. By the definition of tournament, either vw or wv must be an arc of T. Without loss of generality, we assume vw is an arc. If $x \in T$ is any player such that wx is an arc, then vx is also an arc by transitivity. Since vw is an arc, it follows that $s(v) \geqslant s(w) + 1$. In particular, $s(v) \neq s(w)$. Thus, the scores are all different.

(b) If every player in a tournament T has a different score, then the score sequence is $0, 1, \ldots, n-1$ by PAUSE 9. Let v_i be the player with score i. By PAUSE 11, $v_{n-1}v_{n-2} \cdots v_1 v_0$ is a Hamiltonian path. It is unique because in any other listing of vertices, there would be an adjacent pair $v_i v_j$ with $j > i$, implying an arc from v_i to v_j (i.e., v_i beats v_j). However, this is not true since the proof of PAUSE 11 has shown that v_j beats v_i if $j > i$. □