▶ **Problem 12.1-11**

(a) Given that a tree has 100 vertices of degree 1 and 20 of degree 6 and that one-half of the remaining vertices are of degree 4 and the rest of degree 2, determine the number of vertices of degree 2.

(b) Prove that a tree as specified in (a), except that the number of vertices of degree 1 must be less than 80, cannot exist.

Solution. (a) Suppose that the tree T has n vertices. By Theorem 12.1.6, T has $n-1$ edges, so the sum of the degrees is $2(n-1)$. This gives

$$100 + 20(6) + \frac{1}{2}(n-120)(4) + \frac{1}{2}(n-120)(2) = 2(n-1).$$

Thus, $n = 138$. The number of vertices of degree 2 is $\frac{1}{2}(138-120) = 9$.

(b) Suppose that the tree T has n vertices and let k be the number of vertices of degree 1 in T. By assumption, $k < 80$. Then,

$$k + 20(6) + \frac{1}{2}(n-120)(4) + \frac{1}{2}(n-120)(2) = 2(n-1).$$

So $n = 2k-62$. We know that there are at least $k+20$ vertices in T, so $2k-62 \geqslant k+20$. This implies $k \geqslant 82$, which contradicts to the assumption that $k < 80$. This shows that no such a tree exists. □