▶ **Problem 12.2-08**

Determine the number of spanning trees of the complete bipartite graph $K_{2,n}$.

Proof. Let the bipartition sets be $V = \{v_1, v_2\}$ and W, where $|W| = n \geqslant 1$. In any spanning tree of $K_{2,n}$, if two vertices of W are adjacent to both v_1 and v_2, there would be a cycle, which is not allowed. On the other hand, if no vertex of W was adjacent to both v_1 and v_2 in a spanning tree, then the tree would not be connected, another contradiction. Thus, exactly one of the n vertices in W is adjacent to both v_1 and v_2. Say such a vertex w and there are $\binom{n}{1}$ ways to choose the vertex w. We now suppose that k vertices chosen from $W - \{w\}$ are adjacent to v_1 in a spanning tree, and thus $(n-1) - k$ vertices are adjacent to v_2. Therefore, the total number of spanning trees of $K_{2,n}$ is

$$\binom{n}{1} \sum_{k=0}^{n-1} \binom{n-1}{k} = n2^{n-1}.$$

□