▶ **Problem 13.1-20 (c)**

Prove that every planar graph with $V \geqslant 4$ vertices has at least four vertices of degree $d \leqslant 5$.

Proof. We may assume that G is connected and there are just three vertices of degree at most 5. Then $\sum \deg v_i \geqslant 6(V-3)+3 = 6V-15$, where "+3" because G is connected, so there are no vertices of degree 0. Since $\sum \deg v_i = 2E$ must be even and $\sum \deg v_i = 2E \leqslant 6V-12$, there are two cases to consider as follows:

Case 1: $\sum \deg v_i = 2E = 6V-14$. In this case, we note that G must have a vertex of degree 1; otherwise, $\sum \deg v_i \geqslant 6(V-3)+3(2) = 6V-12$. Let G' be the graph obtained from G by deleting the vertex of degree 1 (and the edge incident with it). Clearly, G' is still connected and planar. So, applying theorem 13.1.4 to G', we obtain $\sum \deg v_i \leqslant 6(V-1)-12 = 6V-18$. Also, since G' is obtained from G by removing one vertex and one edge, we have $\sum \deg v_i = (6V-14)-2 = 6V-16$, a contradiction.

Case 2: $\sum \deg v_i = 2E = 6V-12$. In this case, if there is a vertex of degree 1 in G, the same argument as in Case 1 would give a contradiction. Thus, there is a vertex of degree 2; otherwise, $\sum \deg v_i \geqslant 6(V-3)+3(3) = 6V-9$. Let G' be the graph obtained from G by deleting the vertex of degree 2 (and the two edges incident with it). Clearly, G' is planar. So, applying theorem 13.1.4 to G', if G' is connected, we obtain $\sum \deg v_i \leqslant 6(V-1)-12 = 6V-18$. On the other hand, if G' is disconnected, then it contains two components. By the result of Exercise 13.1-19(b), we have $\sum \deg v_i \leqslant 6(V-1)-6(2) = 6V-18$. Also, since G' is obtained from G by removing one vertex and two edges, we have $\sum \deg v_i = (6V-12)-4 = 6V-16$, a contradiction.

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