▶ **Problem 13.2-08 (b) (c)**

(a) Let G be a connected graph with n vertices and n edge. Prove that $\chi(G) \leqslant 3$.

(b) Show that (a) remains true if G has $n+1$ edges.

(c) Does (a) remain true if G has $n+2$ edges? Explain.

Proof. (b) By the same reason of (a), we assume that G has more than 3 vertices and let T be a spanning tree of G. From Theorem 12.1.6, we know that T contains $n-1$ edges. Also, by Exercise 7(c), we know that $\chi(T) = 2$. Since G has $n+1$ edges, $G = T \cup \{e, f\}$ for two edges e and f of G. If e and f have a common vertex v, then giving v a third color in T yields a 3-coloring of G, so $\chi(G) \leqslant 3$. On the other hand, since T contains no circuits, if $e = v_1v_2$ and $f = v_3v_4$ have no vertex in common, then one of edges v_1v_3, v_1v_4, v_2v_3, and v_2v_4 is not in T. If this edge is v_iv_j, then giving v_i and v_j the same third color yields a 3-coloring of G.

(c) No. The complete graph on four vertices has $6 = 4 + 2$ edges, but $\chi(K_4) = 4$. □