▶ **Problem 13.2-13** Let $n \geqslant 4$ be a natural number. Let G be the graph that contains of the union of K_{n-3} and a 5-cycle C, together with all possible edges between the vertices of these graphs. Show that $\chi(G) = n$, yet G does not have K_n as a subgraph.

Proof. By coloring K_{n-3} with $n-3$ colors and C with another three, we obtain an n-coloring of G. Thus, $\chi(G) \leqslant 3$. On the other hand, K_{n-3} requires $n-3$ colors and none of which can be used for C since each vertex of C is adjacent to each vertex of K_{n-3}. Thus, $\chi(G) \geqslant 3$. This shows that the equality follows.

To show that G does not contain K_n as a subgraph. We suppose to the contrary. Then, at least three of the vertices in this subgraph must come from C (since K_{n-3} has only $n-3$ vertices). Thus, C would contain a triangle, contrary to the fact that C is a 5-cycle.. □