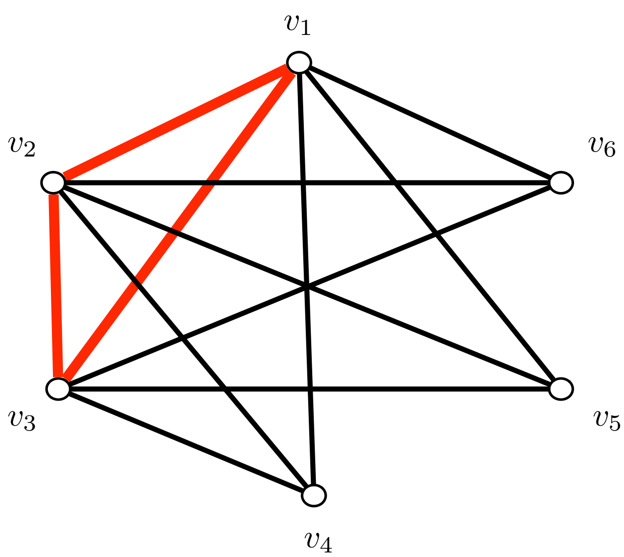
▶ **Problem 9.1-10**

A graph has six vertices, every two of which are joined by an edge. Each edge is colored red or white. Show that the graph contains at least **two** monochromatic triangles.

Proof. Let $G = (V, E)$ be the graph described above, where $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. From Exercise 8(a), we have known that G contains at least one monochromatic triangle. Suppose that the vertices of such a triangle are v_1, v_2 and v_3 (denoted by $\triangle v_1v_2v_3$), and without loss of generality, all the edges of the triangle are colored by red. If there exists any other monochromatic triangle, then we would be done, so we assume that $\triangle v_1v_2v_3$ is the unique monochromatic triangle in G.

Let v_i ($i = 4, 5$ or 6) be any vertex different from v_1, v_2 and v_3. Since at least three edges incident to v_i are colored by the same color, we assume that these edges are v_ix, v_iy and v_iz, where $x, y, z \in V \setminus \{v_i\}$. If one of the edges xy, yz and xz has the same color as v_ix, we would have a second monochromatic triangle; otherwise $\triangle xyz$ forms a second monochromatic triangle. To avoid a second monochromatic triangle, it implies that $\{x, y, z\} = \{v_1, v_2, v_3\}$ and v_ix, v_iy, v_iz are colored by white. Since the above reasoning can be applied to v_4, v_5 and v_6, all the edges in the set $\{v_4v_1, v_4v_2, v_4v_3, v_5v_1, v_5v_2, v_5v_3, v_6v_1, v_6v_2, v_6v_3\}$ are colored by white (see Figure below).



Since $\triangle v_1v_2v_3$ is the only monochromatic triangle in G, one of v_4v_5, v_5v_6, and v_6v_4 must be colored by white, and therefore a second monochromatic triangle is contained in G. □