▶ **Problem 9.2-34**

Suppose that $d_1, d_2, \ldots, d_n$ are the degrees of vertices in some graph. Show that, for any $t < n$,

$$\sum_{i=1}^{t} d_i \leqslant t(t-1) + \sum_{i=t+1}^{n} \min\{t, d_i\}.$$

Remark: The given condition (holding for all $t < n$) and the requirement that $\sum_{i=1}^{n} d_i$ is even are sufficient as well as necessary for the existence of a graph with a prescribed set $d_1, \ldots, d_n$ of degrees.

Proof. Suppose that $G = (V, E)$ is the graph we considered here and let $v_1, v_2, \ldots, v_t$ be vertices with degrees $d_1, d_2, \ldots, d_t$, respectively. Then, the edges incident with v_i for $1 \leqslant i \leqslant t$ are of two types: (I) edges joining two of $v_1, v_2, \ldots, v_t$; and (II) edges joining one of $v_1, v_2, \ldots, v_t$ to a vertex $v \in V \setminus \{v_1, v_2, \ldots, v_t\}$. For each i with $1 \leqslant i \leqslant t$, we write $d_i = d_i' + d_i''$, where d_i' is the number of edges of type (I) incident with v_i, and d_i'' is the number of edges of type (II) incident with v_i. Thus,

$$\sum_{i=1}^{t} d_i = \sum_{i=1}^{t} d_i' + \sum_{i=1}^{t} d_i'' \tag{1}$$

Let H be the subgraph of G induced by the vertices $v_1, v_2, \ldots, v_t$. By Proposition 9.2.5, $\sum_{i=1}^{t} d_i'$ is twice the number of edges in H and this number is at most the number of edges in the complete graph K_t. That is,

$$\sum_{i=1}^{t} d_i' \leqslant 2\binom{t}{2} = t(t-1). \tag{2}$$

Now $\sum_{i=1}^{t} d_i''$ is the sum of all edges of type (II) which is the sum of the number of edges with the form vv_i for $v \in V \setminus \{v_1, v_2, \ldots, v_t\}$ and $1 \leqslant i \leqslant t$. For each v, this number cannot exceed t, nor can it exceed the degree of v. Thus,

$$\sum_{i=1}^{t} d_i'' \leqslant \sum_{i=t+1}^{n} \min\{t, d_i\}. \tag{3}$$

Combining (1), (2), and (3), the result follows.

□