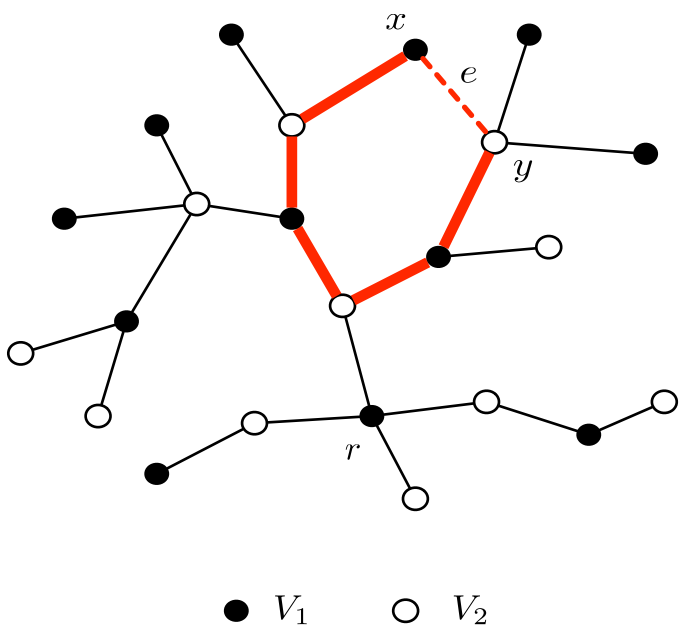
▶ **Problem 10.1-25**

Prove that a graph is bipartite if and only if it contains no odd cycles.

Proof. ($\Leftarrow$) Assume that $G = (V, E)$ is a graph without odd cycle. We will show that G is bipartite by the assignment of colors on vertices. Since a graph is bipartite if all its components are bipartite or trivial, so we may assume that G is connected. Let T be a spanning tree in G, and pick a vertex $r \in T$ to be the root. For each vertex $v \in V$, we denote the unique path from r to v in T by $P_T(r, v)$. Then, we define a bipartition of V in T as follows: $v \in V_1$ if the length of $P_T(r, v)$ is even; and $v \in V_2$ otherwise. We now show that G is bipartite with the same bipartition. Consider an edge $e = (x, y)$ of G and suppose $e \notin T$. Then, the unique path from x to y in T together with e forms a cycle in G (see Figure). Let C be such a cycle. Since vertices along the path from x to y in T alternate between two color classes and C is even by assumption, x and y again lie in different color classes.



($\Rightarrow$) Conversely, if G has a cycle of odd length, we would need at least three colors just for that cycle. Thus, any bipartite graph cannot contain an odd cycle. This complete the proof. □