▶ **Problem 11.4-5 (b)**

Suppose $s_1, s_2, \ldots, s_n$ are the scores in a tournament. Prove that

$$\sum_{i=1}^{n} s_i^2 = \sum_{i=1}^{n} (n-1-s_i)^2$$

and interpret this result in the context of a round-robin tournament.

Proof.

$$\begin{aligned} \sum_{i=1}^{n} (n-1-s_i)^2 &= \sum_{i=1}^{n} \left[(n-1)^2 - 2(n-1)s_i + s_i^2\right] \\ &= \sum_{i=1}^{n} (n-1)^2 - 2(n-1)\sum_{i=1}^{n} s_i + \sum_{i=1}^{n} s_i^2. \end{aligned}$$

Using the fact that $\sum_{i=1}^{n}(n-1)^2 = n(n-1)^2$ and $\sum_{i=1}^{n} s_i = \binom{n}{2} = \frac{1}{2}n(n-1)$ (i.e., the sum of the scores is just the number of arcs, by Proposition 11.2.2) the result follows.

In a tournament with n players, each player plays $n-1$ games, so if a player wins s_i games, he loses $n-1-s_i$ games. The result says that the sum of the squares of all the wins in a tournament equals the sum of the squares of all the losses. □