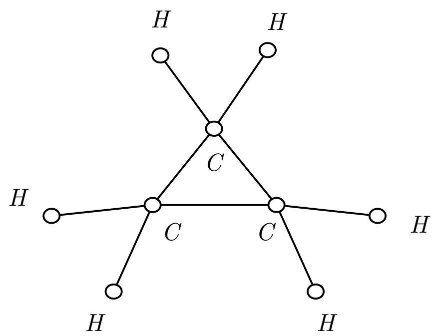
▶ **Problem 12.1-22**

(a) Suppose T is a tree with k vertices labeled C, each of degree at most 4. Enlarge T by adjoining sufficient vertices labeled H so that each vertex C has degree 4 and each vertex H has degree 1. Prove that the number of H vertices adjoined to the graph must be $2k + 2$.

(b) Can you prove (a) without assuming T is a tree?

Proof. (a) Let x be the number of H vertices adjoined. Since T has $k-1$ edges, and one new edge is added for each H, T has $(k-1)+x$ edges. Therefore, $\sum \deg v_i = 2(k-1+x)$. But $\sum \deg v_i = 4k + x$ since each C has degree 4 and each H has degree 1. Therefore, $4k + x = 2k - 2 + 2x$ and $x = 2k + 2$.

(b) The above proof depends on T being a tree. The result is false otherwise. Consider a G which is a 3-cycle and each vertex of G is labeled by C. After enlarging G by sufficient vertices labeled H, the graph is shown as follows.



Here $2k + 2 = 8$, but only six H's are needed.

$\square$