▶ **Problem 13.1-19 (b)**

If G is a planar graph with n connected components, prove that $E \leqslant 3V - 3n$. Deduce that Theorem 13.1.4 holds for arbitrary planar graphs.

Proof. Let $G_1, G_2, \ldots, G_n$ be the connected components of G such that each component G_i has V_i vertices and E_i edges. If $V_i \geqslant 3$, by Theorem 13.1.4 we have $E_i \leqslant 3V_i - 6$. If G_i has two vertices, then $E_i = 1$, so $E_i = 3V_i - 5$. If G_i has one vertex, then $E_i = 0$ and $E_i = 3V_i - 3$. In all cases, $E_i \leqslant 3V_i - 3$, so $E = \sum E_i \leqslant 3 \sum V_i - 3n = 3V - 3n$. Finally, note that if $n \geqslant 2$, then $3V - 3n \leqslant 3n - 6$ so Theorem 13.1.4 holds; while, if $n = 1$, the graph is connected; we established $E \leqslant 3V - 6$ for such a graph in the text. □