▶ Exercise 10-2

Is it true that if $S(G)$ is Hamiltonian, then G is Eulerian?

Solution. Yes, it is true.

Proof. Let $\delta(G)$ and $\Delta(G)$ denote the minimum degree and the maximum degree of vertices of G, respectively. First, we observe that if $S(G)$ is Hamiltonian, then G must be connected. If G has a vertex of degree 1, then so is $S(G)$. Since every Hamiltonian graph has no vertex of degree 1, this implies $\delta(G) \geqslant 2$. On the other hand, if G contains a vertex of degree 3 or more, the $S(G)$ contains a vertex adjacent to at least three vertices of degree 2. Since no Hamiltonian graph has such a vertex, $\Delta(G) \leqslant 2$. So G is a 2-regular connected graph. That is, $G \cong C_n$ (a cycle of length n) for some integer $n \geqslant 3$. Hence, G is Eulerian. □